EDWARD D. MANSFIELD.


ISACUS BARROW S.T.P. REG. MATI A SACRIS COLL. S.S. TRINI. CANTAB. PRÆFEC. NEC NON ACAD. EJUSDEM PROCANC. 1676

J. Mynde sc.

Euclides

EUCLIDE's ELEMENTS;

The whole FIFTEEN BOOKS compendiously Demonstrated: WITH ARCHIMEDES's Theorems of the Sphere and Cylinder Investigated by the *Method of Indivisibles.*

By ISAAC BARROW, D.D. *late Master of* Trinity *College in* Cambridge.

To which is Annex'd,

EUCLIDE's *Data*, and a brief *Treatise of Regular Solids.*

The Whole revis'd with great Care, and some Hundreds of Errors of the former Impression corrected.

By THOMAS HASELDEN, *Teacher of the Mathematicks.*

Καθαρμοὶ ψυχῆς λογικῆς εἰσιν αἱ μαθηματικαὶ ἐπιστῆμαι.

LONDON: Printed for *Daniel Midwinter* and *Aaron Ward* in *Little-Britain*; *Arthur Bettesworth* and *Charles Hitch* in *Pater-noster-row*; and *Thomas Page* and *William Mount* on *Tower-Hill.* 1732.

To the READER.

IF you are desirous, Courteous Reader, to know what I have performed in this Edition of the Elements of Euclide, I shall here explain it to you in short, according to the Nature of the Work. I have endeavour'd to attain two Ends chiefly; the first, to be very perspicuous, and at the same time so very brief, that the Book may not swell to such a Bulk, as may be troublesome to carry about one, in both which I think I have succeeded. Some of a brighter Genius, and endued with greater Skill, may have demonstrated most of these Propositions with more nicety, but perhaps none with more succinctness than I have; especially since I alter'd nothing in the Number and Order of the Author's Propositions; nor presum'd either to take the Liberty of rejecting, as less necessary, any of them, or of reducing some of the easier sort into the Rank of Axioms, as several have done; and among others, that most expert Geometrician A. Tacquetus C. (whom I the more willingly name, because I think it is but civil to acknowledge that I have imitated him in some Points) after whose most accurate Edition I had no Thoughts of attempting any thing of this Nature, 'till I consider'd that this most learned Man thought fit to publish only Eight of Euclide's Books, which he took the pains to explain and embellish, having in a manner rejected and undervalued the other Seven, as less appertaining to the Elements of Geometry. But my Province was originally quite different, not that of writing the Elements of Geometry after what method soever I pleas'd, but of demonstrating, in as few Words as possible I could, the whole Works of Euclide. As

to Four of the Books, viz. *the Seventh, Eighth, Ninth, and Tenth, although they don't so nearly appertain to the Elements of plain and solid Geometry, as the six precedent and the two subsequent, yet none of the more skilful Geometricians can be so ignorant as not to know that they are very useful for Geometrical Matters, not only by reason of the mighty near affinity that is between Arithmetick and Geometry, but also for the Knowledge of both commensurable and incommensurable Magnitudes, so exceeding necessary for the Doctrine of both plain and solid Figures. Now the noble Contemplation of the five regular Bodies that is contained in the three last Books, cannot without great Injustice be pretermitted, since that for the sake thereof our* στοιχειωτὴς, *being a Philosopher of the* Platonic Sect, *is said to have compos'd this universal System of Elements; as* Proclus *lib.* 2. *witnesseth in these Words,* Ὅθεν δὴ καὶ τῆς συμπάσης στοιχειώσεως τέλος προεστήσατο τὴν τῶν καλουμένων Πλατωνικῶν σχημάτων σύστασιν. *Besides, I easily perswaded my self to think, that it would not be unacceptable to any Lover of these Sciences to have, in his Possession the whole* Euclidean *Work, as it is commonly cited and celebrated by all Men: Wherefore I resolv'd to omit no Book or Proposition of those that are found in* P. Herigonius's *Edition, whose Steps I was oblig'd closely to follow, by reason I took a Resolution to make use of most of the Schemes of the said Book, very well foreseeing that Time would not allow me to form new ones, though sometimes I chose rather to do it. For the same Reason I was willing to use for the most part* Euclide's *own Demonstrations, having only express'd them in a more succinct Form, unless perhaps in the Second, Thirteenth, and very few in the Seventh, Eighth, and Ninth Books, in which it seem'd not worth my while to detviae in any Particular from him: Therefore I am not*

without

without good hopes that as to this Part I have in some measure satisfied both my own Intentions, and the Desire of the Studious. As for some certain Problems and Theorems that are added in the Scholions (or short Expositions) either appertaining (by reason of their frequent Use) to the Nature of these Elements, or conducing to the ready Demonstration of those Things that follow, or which do intimate the Reasons of some principal Rules of Practical Geometry, reducing them to their original Fountains, these I say, will not, I hope, make the Book swell to a Size bejond the design'd Proportion.

The other Butt which I levell'd at, is to content the Desires of those who are delighted more with symbolical than verbal Demonstrations. In which Kind, whereas most among us are accustom'd to the Symbols of Gulielmus Oughtredus, *I therefore thought best to make use, for the most part, of his. None hitherto (as I know of) has attempted to interpret and publish* Euclide *after this manner, except* P. Herigonius; *whose Method (tho' indeed most excellent in many things, and very well accomodated for the particular purpose of that most ingenious Man) yet seems in my Opinion to larbour under a double Defect. First, in regard that, altho' of two or more Propositions produced for the Proof of any one Problem or Theorem, the former don't always depend on the latter, yet it don't readily enough appear, either from the order of each or by any other manner, when they agree together, and when not; wherefore for want of the Conjunctions and Adjectives,* ergo, rursus, *&c. many difficulties and occasions of doubt do often arise in reading, especially to those that are Novices. Besides it frequently happens, that the said Method connot avoid superfluous Repetitions, by which the Demonstrations are oftentimes*

times render'd tedious, and ſometimes alſo more intricate; which Faults my Method doth eaſily remedy by the arbitrary mixture of both Words and Signs: Therefore let what has been ſaid, touching the Intention and Method of this little Work, ſuffice. As to the reſt, whoever covets to pleaſe himſelf with what may be ſaid, either in Praiſe of the Mathematicks in general, or of Geometry in particular, or touching the Hiſtory of theſe Sciences, and conſequently of Euclide *himſelf, (who digeſted thoſe Elements) and others* ἐξωτερικοὶ *of that kind, may conſult other Interpreters. Neither will I (as if I were afraid leſt theſe my Endeavours may fall ſhort of being ſatisfactory to all Perſons) alledge as an Excuſe (tho' I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occaſion'd by other Affairs, nor yet the want of requiſite help for theſs Studies, nor ſeveral other things of the like Nature. But what I have here employ'd my Labour and Study in for the Uſe of the ingenious Reader, I wholly ſubmit to his Cenſure and Judgment, to approve if uſeful, or reject if otherwiſe.*

I. B.

Ad amicissimum Virum, I. C. de EUCLIDE contracta, Εὐφημισμός.

FActum bene! didicit Laconice loqui
Senex profundus, & aphorismos induit.
Immensa dudum margo commentarii
Diagramma circuit minutum; utque Insula
Problema breve natabat in vasto mari.
Sed unda jam detumuit; & glossa arctior
Stringit Theoremata: minoris anguli
Lateribus ecce totus Euclides *jacet,*
Inclusus olim velut Homerus in nuce;
Pluteoque sarcina modo qui incubuit, levis
En fit manipulus. Pelle in exigua latet
Ingens Mathesis, matris utero Hercules,
In glande quercus, vel Ithaca Eurus in pila.
Nec mole dum decrescit, usu fit minor;
Quin auctior jam evadit, & cumulatius
Contracta prodest erudita pagina
Sic ubere magis liquor è presso affluit;
Sic pleniori vasa inundat sanguinis
Torrente cordis Systole; sic fusius
Procurrit æquor ex Abylæ angustiis.
Tantilli operis ars tanta referenda unice est
BAROVIANO *nomini, ac solertiæ.*
Sublimis euge mentis ingenium potens!
Cui invium nil, arduum esse nil solet;
Sic usque pergas prospero conamine.
Radiusque multum debeat ac abacus tibi;
Sic crescat indies feracior seges,
Simili colonum germine assiduo beans.
Specimen futuræ messis hic siet labor.
Magnæque famæ illustria hæc præludia.
Juvenis dedit qui tanta, quid dabit senex?

Car. Robotham, *CANTAB.*
Coll. Trin. Sen. Soc.

The

The Explication of the Signs or Characters.

=	Signifies	Equal.
⊏		Greater.
⊐		Lesser.
+		More, or to be added.
−		Less, or to be subtracted.
∽		The Differences, or Excess; Also, that all the Quantities which follow, are to be subtracted, the Signs not being changed.
×		Multiplication, or the Drawing one side of a Rectangle into another. The same is denoted by the Conjunction of Letters; as AB=A x B.
∺		Continued Proportion.
√		The Side or Root of a Square, or Cube, &c.
Q & q		A Square.
C & c		A Cube.
Q. Q.		The Ratio of a square Number to a square Number.

Other Abbreviations of Words, where-ever they occur, the Reader will without trouble understand of himself; saving some few, which, being of less general use, we refer to be explained in their Places, most commonly at the beginning of each Book in which they are used.

The

The First Book of Euclide's Elements.

Definitions.

I. A Point is that which hath no part.

II. A line is a longitude without latitude.

III. The ends, or limits of a line are points.

IV. A right line is that which lies equally betwixt its points.

V. A Superficies is that which hath only longitude and latitude.

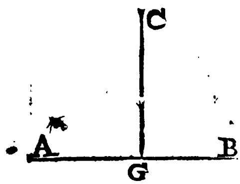
VI The extremes, or limits of a ſuperficies are lines.

VII. A plain ſuperficies is that which lies equally betwixt its lines.

VIII. A plain angle is the inclination of two lines the one to the other, the one touching the other in the ſame plain, yet not lying in the ſame ſtrait line.

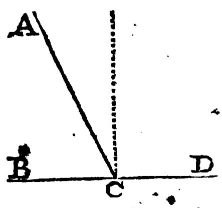
IX. And if the lines which contain the angle, be right lines, it is called a right-lined angle.

 X. When



X When a right-line CG, ſtanding upon a right-line AB, makes the angles on either ſide thereof, CGA, CGB, equal one to the other, then both thoſe equal angles are right-angles; and the right-line CG, which ſtandeth on the other, is termed a Perpendicular to that (AB) whereon it ſtandeth.

Note, *When ſeveral angles meet at the ſame point* (*as at* G) *each particular angle is deſcribed by three letters; whereof the middle letter ſheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right-lines* CG, AG *make at* G, *is called* CGA, *or* AGC.



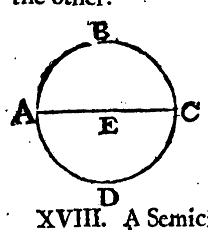
XI. An obtuſe-angle is that which is greater than a right-angle; as ACD.

XII. An acute-angle is that which is leſs than a right-angle; as ACB.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

XV. A Circle is a plain figure contained under one line, which is called a circumference; unto which all lines, drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.



XVI. And that point is called the center of the circle.

XVII A Diameter of a circle is a right-line drawn thro' the center thereof, and ending at the circumference on either ſide, dividing the circle into two equal parts.

XVIII. A Semicircle is a figure which is contained under the diameter and that part of the circumference which is cut off by the diameter.

In the circle EABCD, E *is the center*, AC *the diameter*, ABC *the ſemicircle*.

XIX. Right-lined figures are ſuch as are contained under right-lines.

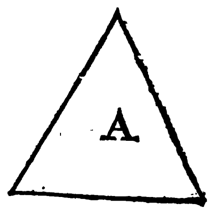
XX. Three

XX. Three-sided or trilateral figures are such as are contained under three right-lines.

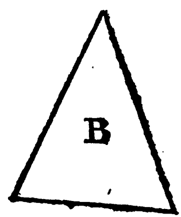
XXI. Four-sided or quadrilateral figures are such as are contained under four right-lines.

XXII. Many-sided figures are such as are contained under more right-lines than four.

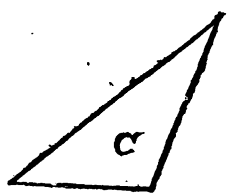
XXIII. Of trilateral figures, that is, an equilateral triangle, which hath three equal sides; as the triangle A.



XXIV. Isosceles is a triangle which hath only two sides equal; as the triangle B.

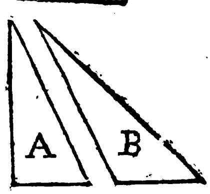


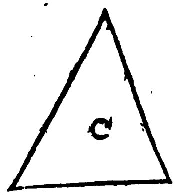
XXV. Scalenum is a triangle whose three sides are all unequal; as C.



XXVI. Of these trilateral figures, a right-angled triangle is that which hath one right-angle; as the triangle A.

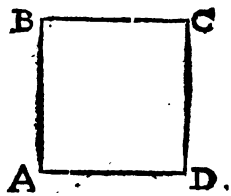
XXVII. An amblygonium, or obtuse-angled triangle, is that which hath one angle obtuse; as B.



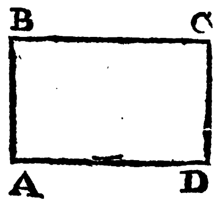


XXVIII. An oxygonium, or acute-angled triangle, is that which hath three acute angles; as C.

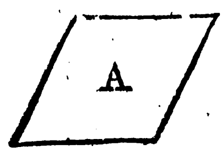
An equiangular, or equal-angled figure is that whereof all the angles are equal. Two figures are equiangular, if the ſeveral angles of the one figure be equal to the ſeveral angles of the other. The ſame is to be underſtood of equilateral figures.



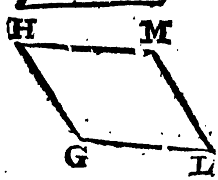
XXIX. Of Quadrilateral, or four-ſided figures, a ſquare is that whoſe ſides are equal, and angles right; as ABCD.



XXX. A Figure on the one part longer, or a long ſquare, is that which hath right angles, but not equal ſides; as ABCD.



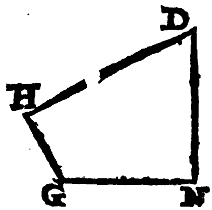
XXXI. A Rhombus, or Diamond-figure, is that which has four equal ſides, but is not right-angled; as A.

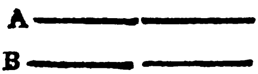


XXXII. A Rhomboides, is that whoſe oppoſite ſides, and oppoſite angles, are equal; but has neither equal nor right angles; as GLMH.

XXXIII. All

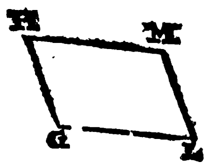
XXXIII. All other quadrilateral figures beſides theſe are called *trapezia*, or tables; as GNDH.



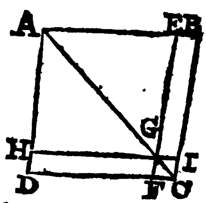


XXXIV. Parallel, or equidiſtant right lines are ſuch, which being in the ſame ſuperficies, if infinitely produced, would never meet; as A and B.

XXXV. A Parallelogram is a quadrilateral figure, whoſe oppoſite ſides are parallel, or equidiſtant; as GLMH.



XXXVI. In a Parallelogram ABCD, when a diameter AC, and two lines EF, HI, parallel to the ſides, cutting the diameter in one and the ſame point G, are drawn, ſo that the Parallelogram be divided by them into four Parallelograms; thoſe two DG, GB, through which the diameter paſſeth not, are called complements; and the other two HE, FI, through which the diameter paſſeth, the Parallelograms ſtanding about the diameter.



A Problem *is, when ſomething is propoſed to be done or effected.*

A Theorem *is, when ſomething is propoſed to be demonſtrated.*

A Corollary *is a Conſectary, or ſome conſequent truth gained from a preceeding demonſtration.*

A Lemma *is the demonſtration of ſome premiſe, whereby the proof of the thing in hand becomes the ſhorter.*

Postulates or Petitions.

1. FRom any given point to any other given point to draw a right-line.

2. To produce a finite right-line, strait forth continually.

3. Upon any center, and at any distance, to describe a circle.

Axioms.

1. THings equal to the same thing, are also equal one to the other.

As A=B=C. Therefore A=C; or therefore all A, B, C, are equal the one to the other.

Note, *When several quantities are joyned the one to the other continually with this mark =, the first quantity is by virtue of this axiom equal to the last, and every one to every one: In which case we often abstain from citing the axiom, for brevity's sake; altho' the force of the consequence depends thereon.*

2. If to equal things you add equal things, the wholes shall be equal.

3. If from equal things you take away equal things, the things remaining will be equal.

4. If to unequal things, you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the same third, or to equal things, are equal one to the other. Understand the same of triple, quadruple, &c.

7. Things which are half of one and the same thing, or of things equal, are equal the one to the other. Conceive the same of subtriple, subquadruple, &c.

8. Things which agree together, are equal one to the other.

The converse of this axiom is true in right lines and angles, but not in figures, unless they be like.

Moreover, magnitudes are said to agree, when the parts of the one being apply'd to the parts of the other, they fill up an equal or the same place.

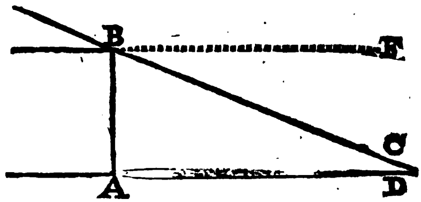
9. Every whole is greater than its part.

10. Two right-lines cannot have one and the same segment (or part) common to them both.

11. Two

11. Two right-lines meeting in the ſame point, if they be both produced, they ſhall neceſſarily cut one the other in that point.

12. All right-angles are equal the one to the other.



13. If a right-line BA, falling on two right-lines, AD, CB, make the internal angles on the ſame ſide, BAD, ABC, leſs than two right-angles, thoſe two right-lines produced ſhall meet on that ſide where the angles are leſs than two right-angles.

14 Two right-lines do not contain a ſpace.

15. If to equal things you add things unequal, the exceſs of the wholes ſhall be equal to the exceſs of the additions

16 If to unequal things equal be added, the exceſs of the wholes ſhall be equal to the exceſs of thoſe which were at firſt.

17. If from equal things unequal things be taken away, the exceſs of the remainders ſhall be equal to the exceſs of what was taken away

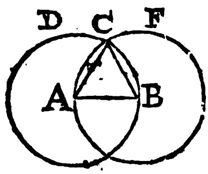
18. If from things unequal things equal be taken away, the exceſs of the Remainders ſhall be equal to the exceſs of the wholes.

19. Every whole is equal to all its parts taken together.

20. If one whole be double to another, and that which is taken away from the firſt be double to that which is taken away from the ſecond, the remainder of the firſt ſhall be double to the remainder of the ſecond.

The Citations are to be underſtood in this manner; *When you meet with two numbers, the firſt ſhews the Propoſition, the ſecond the Book*; as by 4. 1. you are to underſtand the fourth Propoſition of the firſt Book; and ſo of the reſt. *Moreover, ax. denotes Axiom, poſt. Poſtulate, def. Definition, ſch. Scholium, cor. Corollary.*

PROPOSITION I.



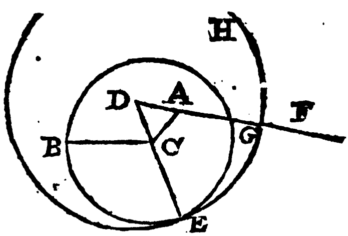
Upon a finite right-line given AB, *to deſcribe an equilateral triangle* ACB.

From the centers A and B, at the diſtance of AB, or BA,
a 3. poſt. *a* deſcribe two circles cutting each other in the point C; from
b 1. poſt. whence *b* draw two right-lines
c 15. def. CA, CB. Then is AC *c*=AB *c*=BC *d*=AC. *e* Where-
d 1. ax. fore the triangle ACB is equilateral. *Which was to be*
e 23. def. *done.*

Scholium.

After the ſame manner upon the line AB may be deſcribed an Iſoſceles triangle, if the diſtances of the equal circles be taken greater or leſs than the line AB.

PROP. II.



From a point given A, *to draw a right-line* AG *equal to a right line given* BC.

a 3. poſt. From the center C, at the diſtance of CB, *a* deſcribe
b 1. poſt. the circle CBE. *b* Join AC; upon which *c* raiſe the
c 1. 1. equilateral triangle ADC. *d* Produce DC to E. From
d 2. poſt. the center D, at the diſtance of DE, deſcribe the circle
e 2. poſt. DEH; and let DA *e* be produced to the point G in the circumference thereof. Then AG=CB.

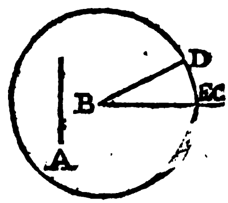
f 15. def. For DG *f*=DE, and DA *g*=DC. Wherefore AG *h*
g conſtr. =CE *k*=BC *l*=AG. *Which was to be done.*
h 3. ax. The putting of the point A within or without the line
k 15. def. BC varies the caſes; but the conſtruction, and the de-
l 1. ax. monſtration, are every where alike.

Schol.

Schol.

The line AG might be taken with a pair of compasses; but the so doing answers to no postulate, as *Proclus* well intimates.

PROP. III.

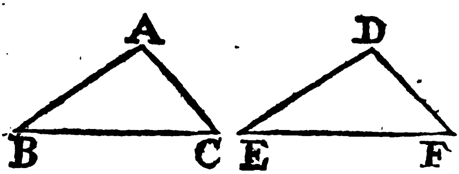


Two right-lines, A *and* BC, *being given, from the greater* BC *to take away the right line* BE *equal to the lesser* A.

From the Point B *a* draw the right line BD = A. The circle described from the center B at the distance of BD shall cut off BE *b* = BD *c* = A *d* = BE. *Which was to be done.*

a 2. 1.
b 15. *def.*
c *constr.*
d 1. *ax.*

PROP. IV.



If two triangles BAC, EDF, *have two sides of the one* BA, AC, *equal to two sides of the other* ED, DF, *each to its correspondent side (that is* BA = ED, *and* AC = DF) *and have the angle* A *equal to the angle* D *contained under the equal right-lines; they shall have the base* BC *equal to the base* EF; *and the triangle* BAC *shall be equal to the triangle* EDF; *and the remaining angles* B, C, *shall be equal to the remaining angles* E, F, *each to each, under which the equal sides are subtended.*

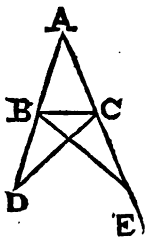
If the point D be apply'd to the point A, and the right-line DE plac'd upon the right line AB, the point E shall fall upon B, because DE *a* = AB, also the right line DF shall fall upon AC, because the angle A *a* = D. Moreover the point F shall fall upon the point C, because AC *a* = DF. Therefore the right-lines EF, BC, shall agree, because they have the same terms, and consequently

a *hyp.*

are

are equal. Wherefore the triangles, BAC, DEF, and the angles B, E, as also the angles C, F, do agree, and are equal. *Which was to be demonstrated.*

PROP. V.



The angles ABC, ACB, *at the base of an Isosceles triangle* ABC, *are equal one to the other; And if the equal sides* AB, AC, *are produced, the angles* CBD, BCE, *under the base, shall be equal one to the other.*

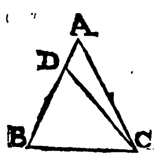
a 3. 1. *a* Take AE=AD; and *b* join CD,
b 1 *post.* and BE.

Because, in the triangles ACD,
c *hyp.* ABE, are AB *c*=AC, and AE *d*=AD, and the angle A
d *constr.* common to them both, *e* therefore is the angle ABE=
e 4. 1. ACD, and the angle AEB *e*=ADC, and the base BE *e*=
f 3. *ax.* CD; also EC *f*=DB. Therefore in the triangles BEC,
g 4. 1. BDC *g* will be the angle ECB=DBC. *Which was to be dem.* Also therefore the angle EBC=DCB, but the
h *before.* angle ABE *h*=ACD; therefore the angle ABC *k*=ACB.
k 3 *ax.* *Which was to be dem.*

Coroll.

Hence, every equilateral triangle is also equiangular.

PROP. VI.



If two angles ABC, ACB *of a triangle* ABC, *be equal the one to the other, the sides* AC, AB, *subtended under the equal angles, shall also be equal one to the other.*

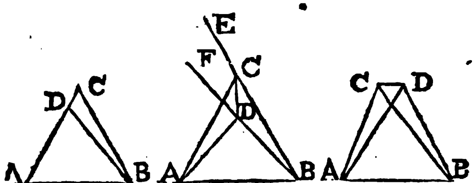
If the sides be not equal, let one be
a 3. 1. bigger than the other, suppose BA ⊏ CA. *a* Make BD=
b 1. *post.* CA, and *b* draw the line CD
c *suppos.* In the triangles DBC, ACB, because BD *c*=CA, and
d *hyp.* the side BC is common, and the angle DBC *d*=ACB, the
e 4. 1. triangles DBC, ACB *e* shall be equal the one to the other,
f 9. *ax.* a part to the whole. *f Which is impossible.*

Coroll

Hence, every equilateral triangle is also equilateral.

PROP.

PROP. VII.



Upon the same right-line AB *two right-lines being drawn* AC, BC, *two other right-lines equal to the former*, AD, BD, *each to each* (viz.) AD=AC, *and* BD=BC) *cannot be drawn from the same points* A, B, *on the same side* C, *to several points, as* C *and* D, *but only to* C.

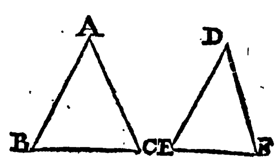
1. *Case* If the point D be set in the line AC, it is plain that AD is *a* not equal to AC. a 9. *ax.*

2. *Case* If the point D be placed within the triangle ACB, then draw the line CD, and produce BDF, and BCE. Now you would have AD=AC, then the angle ADC *b*= ACD; as also, because BD*c*=BC, the angle FDC=*b*ECD, therefore is the angle FDC⊏*d* ACD, that is, the angle FDC⊏ADC. *d Which is impossible.* b 5. 1. c *suppos.* d 9. *ax.*

3. *Case.* If D falls without the triangle ACB, let CD be joined

Again, the angle ACD *e*=ADC, and the angle BCD *e*=BDC. *f* Therefore the angle ACD⊏BDC, *viz.* the angle ADC⊏BDC. *Which is impossible.* Therefore, &c. e 5. 1. f 9. *ax.*

PROP. VIII.



If two triangles ABC, DEF *have two sides* AB, AC, *equal to two sides* DE, DF, *each to each, and the base* BC *equal to the base* EF, *then the angles contained under the equal right lines shall be equal,* viz. A *to* D.

Because, BC *a*=EF, if the base BC be laid on the base EF, *b* they will agree: therefore whereas AB *c*= DE, and AC=DF, the point A will fall on D (for it cannot fall on any other point, by the precedent proposition) and so the sides of the angles A and D are coincident; *d* wherefore those angles are equal. *Which was to be demonstrated.* a *hyp.* b *ax.* 8. c *hyp.* d 8. *ax.*

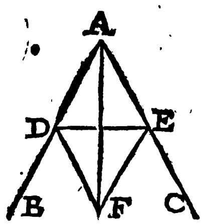
Coroll.

Coroll.

x 4. 1. 1. Hence, triangles mutually equilateral are alſo mutually *x* equiangular.

2. Triangles mutually equilateral *x* are equal one to the other.

PROP. IX.



To biſect, or divide into two equal parts, a right-lined angle given BAC.

a 3. 1. *a* Take AD = to AE, and draw the line DE; upon which *b* make
b 1. 1. an equilateral triangle DFE, draw the right-line AF; it ſhall biſect the angle.

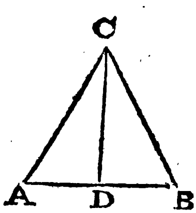
c conſtr. For AD *c* = AE, and the ſide
d 8. 1. AF is common, and the baſe DF *c* = FE. *d* therefore the angle DAF = EAF. *Which was to be done.*

Coroll.

Hence it appears how an angle may be cut into 4, 8, 16, 32, &c. equal parts, to wit, by biſecting each part again.

The method of cutting angles into any equal parts required, by a Rule and Compaſs, is as yet unknown to Geometricians.

PROP. X.



To biſect a right-line given AB.

a 1. 1. Upon the line given AB *a* erect an equilateral triangle ABC; and
b 9. 1. *b* biſect the angle C, with the right line CD. That line ſhall alſo biſect the line given AB.

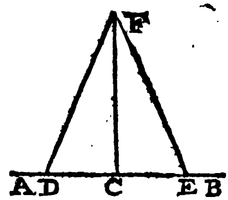
c conſtr. For AC *c* = BC, and the ſide CD is common, and the angle
d 4. 1. ACD *c* = BCD. therefore AD *d* = BD. *Which was to be done.*

The practice of this and the precedent propoſition is eaſily ſhewn by the conſtruction of the 1ſt propoſition of this Book.

PROP.

PROP XI.

From a point C *in a right line given* AB *to erect a right line* CF *at right angles.*



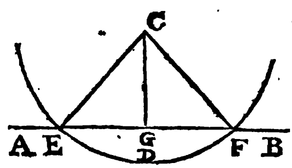
a Take on either side of the point given CD = CE, upon the right-line DE *b* erect an equilateral triangle. draw the line FC, and it will be the perpendicular required. a 3. 1. b 1. 1.

For the triangles DFC, EFC are mutually *c* equilateral; *d* therefore the angle DCF = ECF. *e* therefore FC is perpendicular. *Which was to be done.* c *constr.* d 8. 1. e 10. *def.*

The practice of this and the following is easily performed by the help of a square.

PROP. XII.

Upon an infinite right-line given AB, *from a point given that is not in it, to let fall a perpendicular right line* CG.

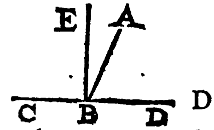


From the center C *a* describe a circle cutting the right-line given AB in the points E and F Then *b* bisect EF in G, and draw the right-line CG, which will be the perpendicular required. a 3. *post.* b 10. 1.

Let the lines CE, CF be drawn. The triangles EGC, FGC are mutually *c* equilateral. *d* therefore the angles EGC, FGC are equal, and by *e* consequence right. *e* Wherefore GC is a perpendicular. *Which was to be done.* c *constr.* d 8. 1 e 10 *def.*

PROP. XIII.

When a right-line AB *standing upon a right-line* CD *maketh angles* ABC, ABD; *it maketh either two right-angles, or two angles equal to two right.*



If the angles ABC, ABD be equal, *a* then they make two right-angles; if unequal, then from the point B *b* let there be erected a perpendicular BE. Because the angle ABC *c* = to a right + ABE, and the angle ABD *d* = to a right − ABE, therefore shall be ABC + ABD *e* = to two right angles + ABE = 2 right angles. *Which was to be demonstrated.* a *def* 10. b 11. 1. c. 19. *ax* d 3 *ax.* e 2 *ax.*

Coroll.

Corollaries.

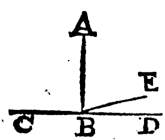
1. Hence, if one angle ABD be right, the other ABC is alſo right; if one acute, the other is obtuſe, and ſo on the contrary.

2. If more right-lines than one ſtand upon the ſame right-line at the ſame point, the angles ſhall be equal to two right.

3. Two right-lines cutting each other make angles equal to four right.

4. All the angles made about one point make four right; as appears by *Coroll.* 2.

PROP. XIV.

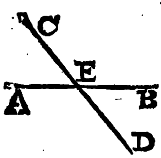


If to any right-line AB, *and a point therein* B, *two right-lines, not drawn from the ſame ſide, do make the angles* ABC, ABD, *on each ſide equal to two right, the lines* CB, BD, *ſhall make one ſtrait line.*

If you deny it, let CB, BE make one right-line; then ſhall be the angle ABC+ABE *a*= two right angles *b*= ABC+ABD. *Which is c abſurd.*

a 13. 1.
b *hyp.*
c 9. *ax.*

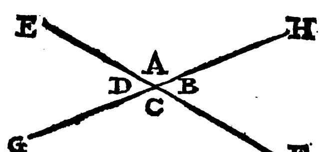
PROP. XV.



If two right lines AB, CD, *cut thro' one another, then are the two angles which are oppoſite,* viz. CEB, AED, *equal one to the other*

a 13. 1. For the angle AEC+CEB *a*= to
b 3. *ax.* two right angles = AEC+AED; *b* therefore CEB=AED. *Which was to be done.*

Schol. 1.



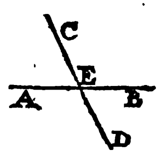
If to any right-line GH, and in it a point A, two right lines being drawn EA, FA, and not taken on the ſame ſide, make the vertical (or oppoſite) angles D and B equal, thoſe

those right-lines EA, FA, do meet directly and make one strait line.

For two right angles are *a* equal to the angle D+A a 13. 1.
b =B+A. *c* Therefore EA, AF, are in a strait line. b 2. *ax.*
Which was to be demonstrated. c. 14. 1.

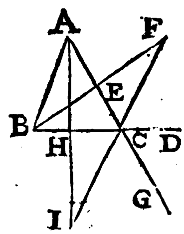
Schol. 2.

If four right-lines EA, EB, EC, ED, proceeding from one point E, make the angles, vertically opposite, equal the one to the other, each two lines, AE,EB, and CE, ED, are placed in one strait line.



For because the angle AEC+AED +CEB+DEB *a* = to four right-angles, therefore the a 4 *c.* 13. 1.
angle AEC+AED *b* =CEB+DEB = to two right an- b *hyp.* &
gles. *c* Therefore CED and AEB are strait lines. *Which* 2 *ax.*
was to be demonstrated. c 14. 1.

PROP. XVI.

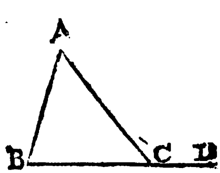


One side BC *of any triangle* ABC *being produc'd, the outward angle* ACD *will be greater than either of the inward and opposite angles*, CAB, CBA.

Let the right-lines AH, BE, *a* bisect the sides AC, BC; from which a 10. 1. &
lines produc'd, take *b* EF=BE, and 1. *post.*
HI, *b* =AH, and join FC, and IC; b 3. 1.
and produce ACG.

Because CE *c* =EA, and EF *c* =EB, and the angle c *constr.*
FEC *d* =BEA, the angle ECF *e* shall be equal to EAB. d 15. 1.
By the like argument is the angle ICH=ABH There- e 4. 1.
fore the whole angle ACD (*f* BCG) *g* is greater than ei- f 15. 1.
ther the angle CAB or ABC *Which was to be demon-* g 9. *ax.*
strated.

PROP XVII.



Two angles of any triangle ABC, *which way soever they be taken, are less than two right angles.*

Let the side BC be produced. Because the angle ACD+ACB
a =two right angles, and the a 13. 1.
angle ACD *b* ⊏ A, *c* therefore A+ACB ⊐ than two b 16. 1.
right c 4. *ax.*

right-angles. After the same manner is the angle B+ACB $\sqsupset$ than two right. Lastly, the side AB being produced, the angle A+B will be also less than two right angles. *Which was to be demonstrated.*

Coroll.

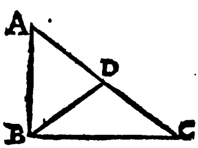
1. Hence it follows that in every triangle wherein one angle is either right or obtuse, the two others are acute angles.

2. If a right-line AE make unequal angles with another right-line DC, one acute AED, the other obtuse AEC, a perpendicular AD, let fall from any point A to the other line CD, shall fall on that side the acute angle is of.

For if AC, drawn on the side of the obtuse angle, be a perpendicular, then in the triangle AEC, shall AEC
* 17. 1. +ACE be greater than two right angles. * *Which is contrary to the precedent prop.*

3. All the angles of an equilateral triangle, and the two angles of an Isosceles triangle that are upon the base, are acute.

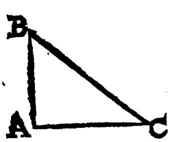
PROP. XVIII.



The greatest side AC *of every triangle* ABC *subtends the greatest angle* ABC.

a 3. 1. From AC *a* take away AD = AB, and join BD. *b* Therefore
b 5. 1. is the angle ADB = ABD. But
c 16. 1. ADB *c* $\sqsubset$ C; therefore is ABD $\sqsubset$ C; *d* therefore the
d 9. ax. whole angle ABC $\sqsubset$ C. After the same manner shall be ABC $\sqsubset$ A. *Which was to be demonstrated.*

PROP. XIX.

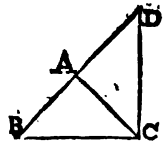


In every triangle ABC, *under the greatest angle* A *is subtended the greatest side* BC.

For if AB be supposed equal to BC, then will be the angle A *a* = C, which
a 5. 1. is contrary to the Hypothesis: and if AB $\sqsubset$ BC, then
b 18. 1. shall be the angle C *b* $\sqsubset$ A, which is against the Hypothesis. Wherefore rather BC $\sqsubset$ AB; and after the same manner BC $\sqsubset$ AC. *Which was to be demonstrated.*

PROP.

PROP. XX.

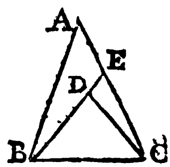


Of every triangle ABC, *two sides* BA, AC, *any way taken, are greater than the side that remains* BC.

Produce the line BA, *a* and take AD=AC, and draw the line DC; *b* then shall the angle D be equal to ACD, *c* therefore is the whole angle BCD $\sqsubset$ D; *d* therefore BD (*e* BA+AC) $\sqsubset$ BC. *Which was to be demonstrated.*

a 3. 1.
b 5. 1.
c 9. ax.
d 19. 1.
e constr. & 2 ax.

PROP. XXI.

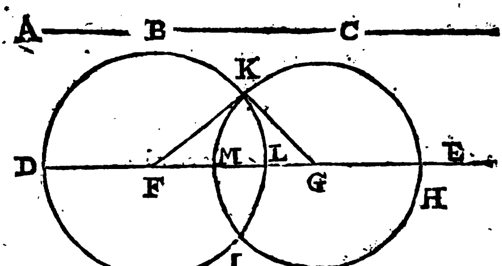


If from the utmost points of one side BC, *of a triangle* ABC, *two right-lines* BD, CD, *be drawn to any point within the triangle, then are both those two lines shorter than the two other sides of the triangle* BA, CA; *but do contain a greater angle,* BDC.

Let BD be produced to E Then is CE+ED *a* $\sqsubset$ CD, add BD common to both, *b* then shall be DB+DE+EC $\sqsubset$ CD+BD Again, BA+AE *a* $\sqsubset$ BE. *b* therefore BA+AC $\sqsubset$ BE+EC. Wherefore 1. BA+AC $\sqsubset$ BD+DC. 2. The angle BDC *c* $\sqsubset$ DEC *c* $\sqsubset$ A. Therefore the angle BDC $\sqsubset$ A. *Which was to be demonstrated.*

a 20. 1.
b 4 ax.
c 16. 1.

PROP. XXII.



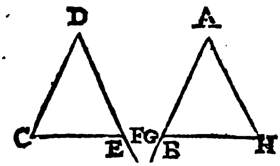
To make a triangle FKG *of three right lines* FK, FG, GK, *which shall be equal to three right-lines given* A, B, C: *Of which it is necessary that any two taken together be longer than the third*

From the infinite line DE *a* take DF, FG, GH, equal to the lines given A, B, C. Then if from the *b* centers F and G at the distances of FD and GH, two circles be

a 3. 1.
b 3. post.

drawn cutting each other in K, and the right lines KF,
c 15. def. KG be joined, the triangle FKG shall be made, *c* whose sides FK, FG, GK, are equal to the three lines DF, FG,
d 1. ax. GH, *d* that is, to the three lines given A, B, C. *Which was to be done.*

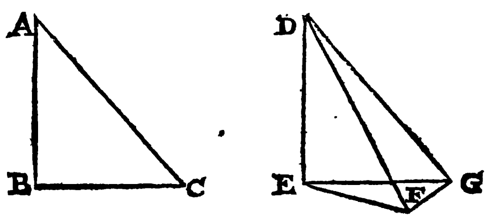
PROP. XXIII.



At a point A *in a right line given* AB, *to make a right-lined angle* A *equal to a right-lined angle given* D.

a 1. post. *a* Draw the right-line CF cutting the sides of the angle given any ways; *b*
b 3. 1. make AG=CD; upon AG *c* raise a triangle equilateral
c 22. 1. to the former CDF, so that AH be equal to DF, and
d 8. 1. GH to CF. then shall you have the angle A *d*=D. *Which was to be done.*

PROP. XXIV.

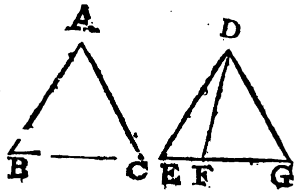


If two triangles ABC, DEF *have two sides of the one triangle* AB, AC *equal to two sides of the other triangle* DE, DF, *each to other, and have the angle* A *greater than the angle* EDF *contained under the equal right-lines, they shall have also the base* BC *greater than the base* EF.

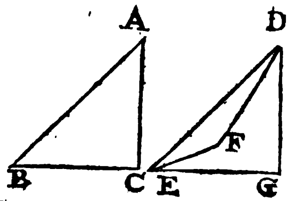
a Let the angle EDG be made equal to A, and the side
a 23. 1. DG *b*=DF *c*=AC; and let EG, and FG be joined.
b 3. 1. 1. *Case.* If EG falls above EF; Because AB *d*=DE,
c hyp. and AC *e*=DG, and the angle A *e*=EDG, *f* therefore
d hyp is BC=EG. But because DF *e*=DG, *g* therefore is the
e constr. angle DFG=DGF; *h* therefore is the angle DFG ⊏
f 4. 1. EGF, and by consequence the angle EFG, *h* ⊏ EGF. *k*
g 5. 1. wherefore EG (BC) ⊏ EF.
h 9. ax.
k 19. 1.

2. Case

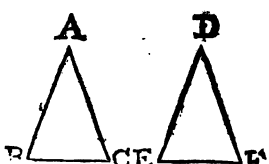
2. *Case.* If the base EF coincides with the base EG, *l* it is evident that EG (BC) $\sqsubset$ EF. l 19. ax.



3. *Case* If EG falls below EF, then because DG + GE *m* $\sqsubset$ DF + FE, if from both be taken away DG, DF which are equal; EG (BC) remains *n* $\sqsubset$ EF. *Which was to be dem.* m 21. 1. n 5. ax.



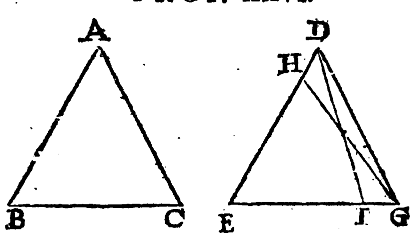
PROP. XXV.



If two triangles ABC, DEF, *have two sides* AB, AC, *equal to two sides* DE, DF, *each to other, and have the base* BC *greater than the base* EF, *they shall also have the angle* A *contained under the equal right lines greater than the angle* D.

For if the angle A be said to be equal to D, *a* then is the base BC = EF, which is against the Hypothesis. If it be said the Angle A $\sqsupset$ D, then *b* will be BC $\sqsupset$ EF, which is also against the Hypothesis. Therefore BC $\sqsubset$ EF. *Which was to be dem.* a 4. 1. b 24. 1.

PROP. XXVI.



If two triangles BAC, EDG, *have two angles of the one* B, C, *equal to two angles of the other* E, DGE, *each to his correspondent angle, and have also one side of the one equal to one side of the other, either that side which lyeth betwixt the*

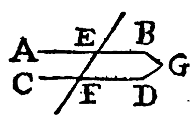
equal angles, or that which is subtended under one of the equal angles; the other sides also of the one shall be equal to the other sides of the other, each to his correspondent side, and the other angle of the one, shall be equal to the other angle of the other.

1. *Hypothesis.* Let BC be equal to EG, which are the sides that lie between the equal angles. Then I say BA=ED, and AC=DG, and the angle A = EDG.
a 3. 1. For if it be said that ED $\sqsubset$ BA, then *a* let EH be made equal to BA, and let the line GH be drawn

b *constr.* Because AB *b* = HE, and BC *c* = EG, and the angle
c *hyp.* B *c*=E, therefore shall be the angle EGH *d* = C *e* =
d 4 1. DGE. *f Which is absurd,* therefore AB=ED. After
e *hyp.* the same manner AC may be proved eqnal to DG, *d* then
f 9. *ax.* will the angle A be equal to EDG.

2. *Hyp.* Let AB be equal to DE, then I say BC = EG, and AC = DG, and the angle A=EDG. For if
g *hyp.* EG be greater than BC make EI= BC, and join
h *constr.* DI. Now because AB *g* =DE, and BC *h* = EI, and
k 4. 1. the angle B *g* =E; therefore will be the angle EID *k*
l *hyp.* = C *l* =EGD *m Which is absurd.* Therefore is BC=
m 16. 1. EG, and so as before, AC = DG, and the angle A = EDG. *Which was to be dem.*

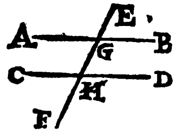
PROP. XXVII.



If a right line EF; *falling upon two right lines* AB, CD, *makes the alternate angles* AEF, DFE, *equal the one to the other, then are the right lines* AB, CD, *parallel*

If AB, CD be said not to be parallel, produce them till they meet in G, which being supposed, the outward
a 16. 1. angle AEF will be *a* greater than the inward angle DFE, to which it was equal by Hypothesis *Which things are repugnant.*

PROP. XXVIII.



If a right line EF, *falling upon two right lines,* AB, CD, *makes the outward angle* AGE *of the one line equal to* CHG *the inward and opposite angle of the other on the same side, or make the inward angles on the same side,* AGH, CHG, *equal to two right angles, then are the right lines* AB, CD, *parallel.*

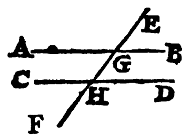
Hyp.

Hyp. 1. Becauſe by Hypotheſis the angle AGE $=$ CHG, *a* therefore are BGH, CHG, the alternate angles equal; And *b* therefore are AB and CD parallel. a 15. 1. b 27. 1.

Hyp. 2. Becauſe by Hypotheſis the angle AGH $+$ CHG $=$ to two right, *a* $=$ AGH $+$ BGH, *b* ſhall be the angle CHG $=$ BGH; and *c* therefore AB, CD, are parallel. *Which was to be demonſtrated.* a 13. 1. b 3. *ax.* c 27. 1.

PROP. XXIX.

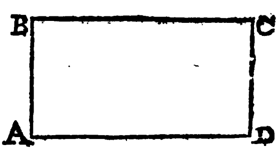
If a right line EF *falls upon two parallels,* AB, CD, *it will make both the alternate angles* DHG, AGH, *equal each to other, and the outward angle* BGE *equal to the inward and oppoſite angle on the ſame ſide* DHG, *as alſo the inward angles on the ſame ſide* AGH, CHG, *equal to two right angles.*



It is evident, that AGH $+$ CHG $=$ two right angles; *a* otherwiſe AB, CD, would not be parallel, which is contrary to the Hypotheſis: But moreover the angle DHG $+$ CHG *b* $=$ two right; therefore is DHG *c* $=$ AGH *d* $=$ BGE. *Which was to be dem.* a 13. *ax.* b 13. 1. c 3 *ax.* d 15. 1.

Coroll.

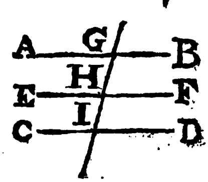
Hence it follows that every parallelogram AC having one angle right A, the reſt are alſo right.



For A $+$ B *a* $=$ two right angles Therefore, whereas A is right, *b* B muſt be alſo right. By the ſame argument are C and D right angles. a 29. 1. b 3. *ax.*

PROP. XXX.

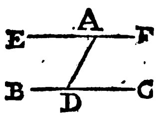
Right lines (AB, CD) *parallel to one and the ſame right line* EF, *are alſo parallel the one to the other*



Let GI cut the three right lines given any ways. Then becauſe AB, EF, are parallel, the angle AGI will be *a* $=$ EHI. Alſo becauſe a 29. 1.

a 29. 1. cause CD and EF are parallel, the angle EHI will be
b 1. ax. a = DIG. b Therefore the angle AGI = DIG. c whence
c 27. 1. AB and CD are parallel. *Which was to be demonstrated.*

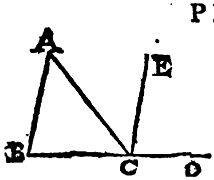
PROP. XXXI.



From a point given A *to draw a right line* AE, *parallel to a right line given* BC.

From the point A draw a right line BD to any point of the given right
a 23. 1. line; with which at the point thereof a A make an an-
b 27. 1. gle DAE = ADC. b then will AE and BC be parallel.
Which was to be done.

PROP XXXII.



Of any triangle ABC *one side* BC *being drawn out, the outward angle* ACD *shall be equal to the two inward opposite angles* A, B, *and the three inward angles of the triangle* A, B, ACB, *shall be equal to two right angles.*

a 31. 1. From C a draw CE parallel to BA Then is the
b 29. 1. angle A b = ACE, and the angle B b = ECD. There-
c 2. ax fore A + B c = ACE + ECD d = ACD. *Which was to be*
d 19 ax. *demonstrated.*
e 13. 1. I affirm ACD + ACB e = two right angles; f there-
f 1. ax. fore A + B + ACB = two right angles. *Which was to be demonstrated.*

Coroll.

1. The three angles of any triangle taken together are equal to the three angles of any other triangle taken together From whence it follows,

2. That if in one triangle, two angles (taken severally, or together) be equal to two angles of another triangle (taken severally, or together) then is the remaining angle of the one equal to the remaining angle of the other In like manner, if two triangles have one angle of the one equal to one of the other, then is the sum of the remaining angles of the one triangle equal to the sum of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right. Likewise, that angle in a triangle which is equal to the other two, is it self a right angle.

4. When in an Isosceles the angle made by the equal sides is right, the other two upon the base are each of them half a right angle.

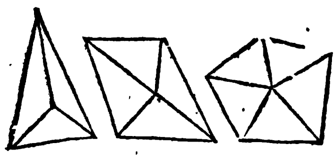
5. An

5. An angle of an equilateral triangle makes two third parts of a right angle. For one third of two right angles is equal to two thirds of one.

Schol.

By the help of this propoſition you may know how many right angles the inward and outward angles of a right lined figure make; as may appear by theſe two following Theorems.

THEOREM I.



All the angles of a right lined figure do together make twice as many right angles, abating four, as there are ſides of the figure

From any point within the figure let right lines be drawn to all the angles of the figure, which ſhall reſolve the figure into as many triangles as there are ſides of the figure. Wherefore, whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are ſides. But the angles about the ſaid point within the figure make up four right; therefore, if from the angles of all the triangles you take away the angles which are about the ſaid point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, abating four, as there are ſides of the figure. *Which was to be demonſtrated.*

Coroll.

Hence all right-lined figures of the ſame ſpecies have the ſums of their angles equal

THEOREM II.

All the outward angles of any right-lined figure, taken together, make up four right angles.

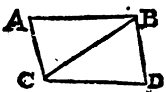
For every inward angle of a figure, with the outward angle of the ſame, make two right angles; therefore all the inward angles, together with all the outward, make twice as many right angles as there are ſides of the figure: but (as has been juſt ſhewn) all the

inward angles, with four right, make twice as many right as there as sides of the figure; therefore the outward angles are equal to four right angles. *Which was to be demonstrated.*

Coroll.

All right-lined figures, of whatsoever species have, the sums of their outward angles equal

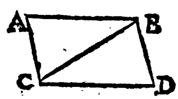
PROP. XXXIII.



If two equal and parallel lines AB, CD, *be joyned together with two other right lines,* AC, BD, *then are those lines also equal and parallel.*

Draw a line from C to B. Now because AB and CD
a 29. 1. are parallel, and the angle ABC *a* = BCD; and also by hypothesis AB = CD, and the side CB common,
b 4. 1. therefore is AC *b* = BD, and the angle ACB *b* = DBC
c 27. 1. *c* whence also AC, BD, are parallel.

PROP. XXXIV.



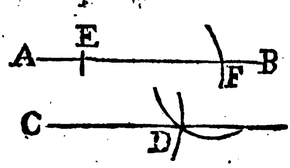
In parallelograms, as ABDC, *the opposite sides* AB, CD, *and* AC, BD, *are equal each to other; and the opposite angles* A, D, *and* ABD, ACD, *are also equal; and the diameter* BC *bisects the same.*

a hyp. Because AB, CD, *a* are parallel, *b* therefore is the angle ABC = BCD. Also because AC, BD, are *a* pa-
b 29. 1 rallel, *b* therefore is the angle ACB = CBD; *c* there-
c 2 ax. fore the whole angle ACD = ABD. After the same manner is A = D. Moreover because the angles ABC, ACB, lie at each end of the side CB, and are equal to
d 26. 1. BCD, CBD, *d* therefore is AC = BD and AB *d* = CD, and so the triangle ABC = CBD. *Which was to be demonstrated.*

Schol.

Every four-sided figure ABDC, *having the opposite sides equal, is a parallelogram*

a 27. 1. For by 8. 1. the angle ABC = BCD; *a* wherefore AB, CD, are parallel In like manner is the angle BCA = CBD; *a* wherefore AC, BD, are also parallel.
b 35 def. 1. *b* Therefore ABCD is a parallelogram. *Which was to be demonstrated.*

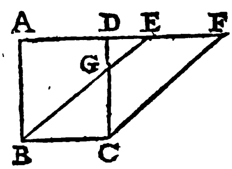


From hence we may more expeditiously draw a parallel CD to a right line given AB, thro' a point assigned C.

Take

Take in the line AB any point, as E. From the centers E and C at any distance draw two equal circles EF, CD. From the center F with the distance EC draw a circle FD, which shall cut the former circle CD in the point D. Then shall the line drawn CD be parallel to AB, for, as it was before demonstrated, CEFD is a parallelogram.

PROP. XXXV.



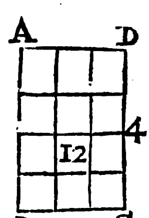
Parallelograms, BCDA, BCFE, *which stand upon the same base* BC, *and between the same parallels* AF, BC, *are equal one to the other.*

For AD *a*=BC *a*=EF, add DE common to both, *b* then is AE=DF. But also AB *a* =DC, and the angle A *c* =CDF. *d* Therefore is the triangle ABE=DCF. take away DGE common to both triangles, *e* then is the Trapezium ABGD=EGCF, add BGC common to both, *f* then is the parallelogram ABCD=EBCF. *Which was to be demonstrated.*

a 34. 1.
b 2. *ax.*
c 29, 1.
d 4, 1.
e 3. *ax.*
f 2. *ax.*

The demonstration of any other cases, is not unlike, but much more plain and easy.

Schol.



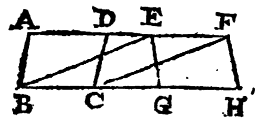
If the side AB of a right angled parallelogram ABCD be conceived to be carried along perpendicularly thro' the whole line BC, or BC thro' the whole line AB, the area or content of the rectangle ABCD shall be produc'd by that motion. Hence a rectangle is said to be made by the drawing or multiplication of two contiguous sides For example; let AB be supposed four foot, and BC three: draw three into four, there will be produced twelve square feet for the area of the rectangle.

This being supposed, the dimension of any parallelogram (*EBCF) is found out by this theorem. For the area thereof is produced from the altitude BA drawn into the base BC. For the area of the rectangle AC = parallelogram EBCF, is made by the drawing of BA into BC, therefore, &c.

* *See the fig. of* Prop 35.

PROP.

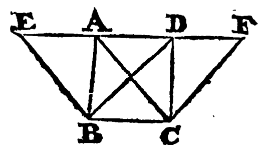
PROP. XXXVI.



Parallelograms BCDA, GHFE, *standing upon equal bases* BC, GH, *and betwixt the same parallels* AF, BH, *are equal one to the other.*

a hyp. Draw BE and CF, Because BC $a=$ GH $b=$ EF, c
b 34. 1. therefore is BCFE a parallelogram. Whence the pa-
c 33. 1. rallelogram BCDA $d=$ BCFE $d=$ GHFE. *Which was*
d 35. 1. *to be demonstrated.*

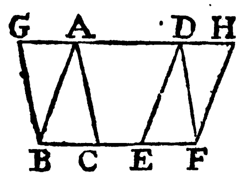
PROP. XXXVII.



Triangles, BCA, BCD, *standing upon the same base* BC, *and between the same parallels* BC, EF, *are equal the one to the other.*

a 31. 1. a Draw BE parallel to
b 34. 1. CA, a and CF parallel to BD. Then is the triangle
c 35. 1. BCA $b=$ half Pgr. BCAE $c=$ half BDFC $b=$ BCD.
and 7. ax. *Which was to be demonstrated.*

PROP. XXXVIII.

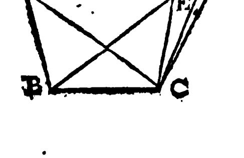


Triangles, BCA, EFD, *set upon equal bases* BC, EF, *and between the same parallels* GH, BF, *are equal the one to the other.*

Draw BG parallel to CA, and FH parallel to ED

a 34. 1. Then is triangle BCA $a=$ half Pgr. BCAG $b=$ half
b 36. 1. EDHF $c=$ EFD. *Which was to be demonstrated.*
and 7. ax. *Schol.*
c 34. 1. If the base BC be greater than EF, then is the tri-
angle BAC $\sqsubset$ EDF, and so on the contrary.

PROP. XXXIX.

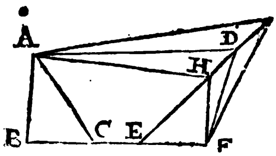


Equal triangles BCA, BCD, *standing on the same base* BC, *and on the same side are also between the same parallels* AD, BC.

If

If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF *a* a 37. 1.
=CBA *b* =CBD. *c Which is absurd.* b *hyp.*

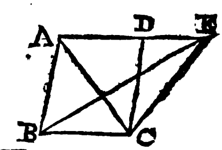
PROP. XL. c 9. *ax.*



Equal triangles BCA, EFD, *standing upon equal bases* BC, EF, *and on the same side, are betwixt the same parallels.*

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle a 38. 1.
EFH *a*=BCA *b*=EFD *c Which is absurd.* b *hyp.*

PROP. XLI. c 9. *ax.*



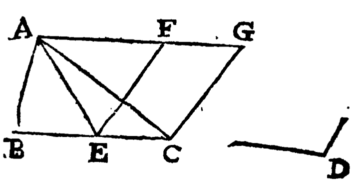
If a Pgr. ABCD *have the same base* BC *with the triangle* BCE, *and be between the same parallels* AE, BC, *then is the Pgr.* ABCD *double to the triangle* BCE.

Let the line AC be drawn. Then is the triangle BCA *a* =BCE, therefore is the a 37. 1.
Pgr. ABCD *b* =2BCA *c* =2BCE. *Which was to be demonstrated.* b 34. 1.
c 6. *ax.*

Schol.

From hence may the area of any triangle BCE be found, for whereas the area of the Pgr. ABCD is produced by the altitude drawn into the base, therefore shall the area of a triangle be produced by half the altitude drawn into the base, or half the base drawn into the altitude; thus, if the base BC be 8, and the altitude 7, then is the area of the triangle BCE 28.

PROP. XLII.

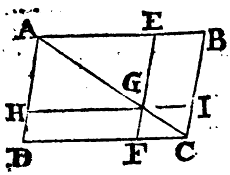


To make a Pgr ECGF *equal to a triangle given* ABC *in an angle equal to a right-lined angle given* D.

Through

a 31. 1. Through A *a* draw AG parallel to BC, *b* make the
b 23. 1. angle BCG=D, *c* bisect the base BC in E, and draw
c 10. 1. EF parallel to CG, then is the problem resolved.
For AE being drawn, the angle ECG is equal to D
d 38. 1. by construction, and the triangle BAC *d* =2 AEC *e* =
e 41. 1. Pgr. ECGF. *Which was to be done.*

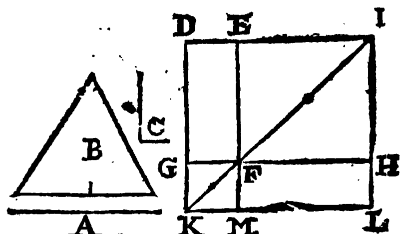
PROP. XLIII.



In every Pgr. ABCD, *the complements* DG, GB, *of those Pgrs.* HE, FI, *which stand about the diameter, are equal one to the other.*

For the triangle ACD *a* = ACB, and the triangle AGH
a 34. 1. *a* = AGE, and the triangle GCF *a* = GCI. *b* There-
b 3. *ax.* fore the Pgr. DG=BG. *Which was to be demonstrated.*

PROP. XLIV.

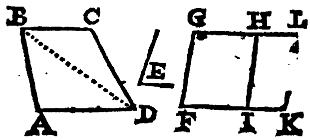


To a given right-line A, *to apply a parallelogram* FL, *equal to a given triangle* B, *in a given angle* C.

a Make a Pgr. FD equal to the triangle B, so that
a 42. 1. the angle GFE may be equal to C. Produce GF till FH be equal to the line given A. Through H *b* draw IL parallel to EF, which let DE produced meet in I, let DG produced meet with a right line drawn from I
b 31. 1. through F in the point K, thro' K *b* draw KL parallel to GH, which let EF drawn out meet at M, and IH at L. Then shall FL be the Pgr. required.
c 43. 1. For the Pgr FL *c*=FD=B, *d* and the angle MFH=
d 15. 1. GFE =C. *Which was to be done.*

PROP.

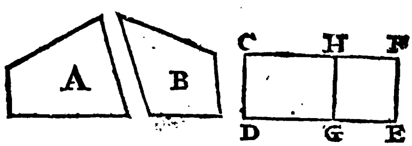
PROP. XLV.



Upon a right line given FG, *and in a given angle* E, *to make a Pgr.* F L, *equal to a right lined figure given* ABCD.

Resolve the right-lined figure given into two triangles BAD, BCD, then *a* make a Pgr. FH = BAD, so that a 44. 1.
the angle F may be equal to E. FI being produced, *a* make on HI the Pgr. IL = BCD. Then is the Pgr. b 19. *ax.*
FL *b* = FH + I L *c* = ABCD. *Which was to be done.* c *constr.*

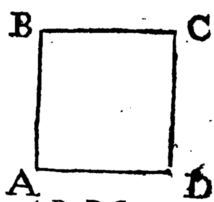
Schol.



Hence is easily found the excess, HE, whereby any right-lined figure, A, exceeds a less right-lined figure, B; namely, if to some right-line, CD, be applied the Pgr. DF = A, and DH = B.

PROP. XLVI.

Upon a right line given AD *to describe a square* AC.



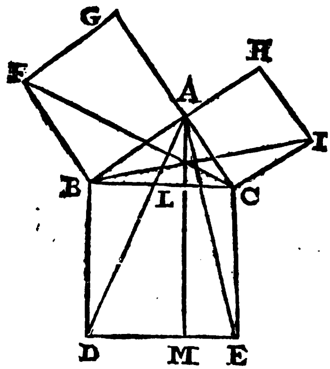
a Erect two perpendiculars AB, DC, *b* equal to the line given AD; then join BC, and the thing required is done. a 11. 1. b 3. 1.

For, whereas the Angle A + D *c* = two right, *d* therefore are AB, DC parallel. c *constr.* d 28. 1.
But they are also *e* equal; *f* therefore AD, BC are both parallel and equal; therefore the figure AC is a Pgr and equilateral. Moreover the angles are all right, *g* because one A, is right; *h* therefore AC is a square. *Which was to be done.* e *constr.* f 33. 1. g *cor.* 29. 1. h 29. *def.*

After

After the same manner you may easily describe a rectangle contained under two right lines given.

PROP. XLVII.



In right-angled triangles BAC, *the square* BE, *which is made on the side* BC *that subtends the right angle* BAC, *is equal to both the squares* BG, CH, *which are made on the sides* AB, AC, *containing the right angle.*

Join AE, and AD; and draw AM parallel to CE.

a 12. *ax.* Because the angle DBC *a*=FBA, add the angle ABC common to them both; then is the angle ABD=
b 29 *def.* FBC. Moreover, AB *b*=FB, and BD *b*=BC; *c* there-
c 4. 1. fore is the triangle ABD=FBC. But the Pgr. BM *d*=
d 41. 1. 2 ABD, and the Pgr. *d* BG= 2 FBC (for GAC is one
e 6. *ax.* right line by Hypothesis, and 14. 1) *e* therefore is the Pgr. BM=BG. By the same way of argument is the
f 2. *ax.* Pgr CM=CH. Therefore is the whole BE=*f* BG+CH. *Which was to be demonstrated.*

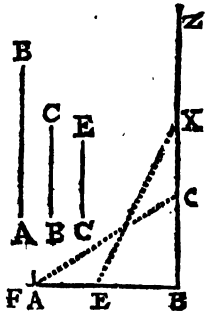
Schol.

This most excellent and useful theorem hath deserved the title of *Pythagoras* his theorem, because he was the inventor of it. By the help of which the addition and subtraction of squares are performed; to which purpose serve the two following problems.

PRO-

PROBLEM I.

To make one square equal to any number of squares given. Andr. Tacq.

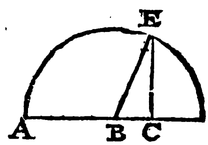


Let three squares be given, whereof the sides are AB, BC, CE. *a* Make the right angle FBZ, having the sides infinite; and on them transfer BA and BC; join AC. then is ACq *b* = ABq + BCq. Then transfer AC from B to X, and CE the third side given from B to E; join EX. *b* Then is EXq = EBq (CEq) + BXq (ACq) *c* = CEq + ABq + BCq. *Which was to be done.*

a 11. 1.
b. 47. 1.
c 2. ax.

PROBLEM II.

Two unequal right lines being given AB, BC, *to make a square equal to the difference of the two squares of the given lines* AB, BC.

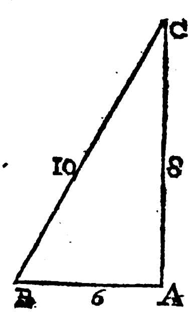


From the center B, at the distance of BA, describe a circle; and from the point C erect a perpendicular CE meeting with the circumference in E; and draw BE. *a* Then is BEq (BAq) = BCq + CEq. *b* Therefore BAq—BCq=CEq. *Which was to be done.*

a 47. 1.
b 3. ax.

PROBLEM III.

Any two sides of a right angled triangle ABC, *being known, to find out the third.*

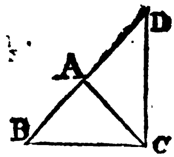


Let the sides AB, AC, encompassing the right angle, be, the one 6 foot, the other 8. Therefore, whereas ACq + ABq = 64 + 36 = 100 = BCq, thence is BC = $\sqrt{100}$ = 10. 47. 1.

Now, let the sides AB, BC, be known, the one 6 foot, the other 10. Therefore since BCq — ABq = 100—36=64=ACq, thence is AC = $\sqrt{64}$ = 8. *Which was to be done* 47. 1.

PROP.

PROP. XLVIII.



If the square made upon one side BC *of a triangle be equal to the squares made on the other sides of the triangle* AB, AC, *then the angle* BAC *comprehended under the two other sides of the triangle* AB, AC, *is a right angle.*

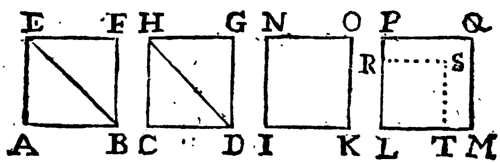
Perpendicular to AC draw AD = AB, and join CD.

a 47. 1. Now is *a* CDq = ADq + ACq = ABq + ACq = BCq.
* *See the following theor.* * Therefore is CD = BC. And therefore the triangles CAB, CAD, are mutually equilateral. Wherefore the
b 8. 1. angle CAB *b* = CAD *c* = a right angle. *Which was to*
c *constr.* *be demonstrated.*

Schol.

We assumed in the demonstration of the last Proposition, *CD* = *BC*, because *CD*q was equal to *BC*q: Our assumption we prove by the following theorem.

THEOREM.



The squares AF, CG of equal right lines AB, CD, are equal one to the other: And the sides IK, LM, of equal squares NK, PM, are equal one to the other.

1. *Hypothesis.* Draw the diameters *EB*, *HD*. Then
a 34. 1. it is evident that *AF* is *a* equal to the triangle *EAB*
b 4. 1. & twice taken, and *b* equal to the triangle *HCD* twice
6. *ax.* taken, and equal to *a* *CG*. *Which was to be demonstrated.*

2. *Hyp* If it may be, let LM be greater than IK.
a 46 1. Make LT = IK, and let LS *a* = LTq. Therefore is
b 1. *part,* LS *b* = NK *c* = LQ. *d Which is absurd.*
c *hyp.*

Coroll.

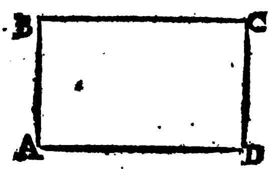
d 9. *ax.* After the same manner any rectangles equilateral one to another, are demonstrated to be also equal.

The End of the first Book.

The

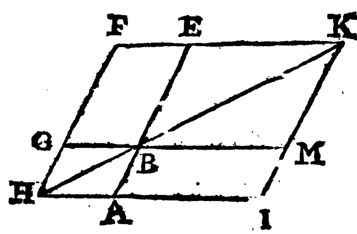
The Second Book OF EUCLIDE'S ELEMENTS.

Definitions.



I. EVery right angled Parallelogram ABCD, is said to be contained under two right lines AB, AD, comprehending a right angle.

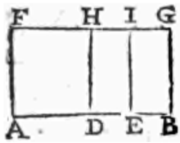
Therefore when you meet with such as these, the rectangle under BA, AD, *or more briefly the rectangle* BAD, *or* BA x AD (or ZA, for Z x A) *the rectangle meant is that which is contained under the right lines* BA, AD, *set at right angles.*



II. In every Pgr. FHIK, any one of those parallelograms which are about the diameter, together with

its two complements is called a Gnomon. *As the Pgr.* FB + BI + GA (EHM) *is a Gnomon; and likewise the Pgr.* FB + BI + EM (GKA) *is a Gnomon.*

PROP. I.



If two right lines AF, AB, *are given, and one of them* AB *divided into as many parts or segments as you please; the rectangle comprehended under the two whole right lines* AB, AF, *shall be equal to all the rectangles contained under the whole line* AF, *and the several segments,* AD, DE, EB.

a 11. 1. *a* Set AF perpendicular to AB. Thro' F *a* draw an infinite line FG perpendicular to AF. From the points D, E, B, erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is

b 19 *ax* 1. *b* equal to the rectangles AH, DI, EG, that is (be-

c 34. 1. cause DH, EI, AF, *c* are equal) to the rectangles under AF, AD, under AF, DE, under AF, EB. *Which was to be demonstrated.*

Schol.

If two right lines given are both divided into how many parts soever, one whole multiplied into the other shall bring out the same product, as the parts of one multiplied into the parts of the other.

For let Z be = A + B + C, and Y = D + E; then,

a 1. 2. because DZ *a* = DA + DB + DC, and EZ *a* = EA +

b 2. *ax.* EB + EC, and YZ *a* = DZ + EZ, *b* shall ZY be = DA + DB + DC + EA + EB + EC. *Which was to be demonstrated.*

From hence we have a method of multiplying compound lines into compound ones. For if the rectangles of all the parts be taken, their sum shall be equal to the rectangle of the wholes.

But whensoever in the multiplication of lines into themselves you meet with these signs — intermingled with these +, you must also have particular regard to the signs. For of + multiplied into — ariseth — ; but of — into — ariseth +. *ex. gr.* let + A be multiplied into B — C; then because + A is not affirmed of all B, but only of that part of it, whereby it exceeds C, therefore AC must remain denied; so that the product will be AB —

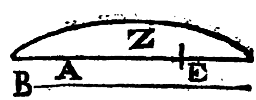
* 1 2. AC. Or thus; because B consists of the parts C and B — C, * thence AB = AC + A x B — C, take away AC from both. then AB — AC = A x B — C. In like manner,

ther, if $-A$ be to be multiplied into $B-C$, then since by virtue of the sign $-$, A is not denied of all B, but only of so much as it exceeds C, therefore AC must remain affirmed, whence the product will be $-AB+AC$. Or thus; because $AB = AC + A \times \overline{B-C}$; take away all from both sides, and there will be $-AB = -AC - A \times \overline{B-C}$; add AC to both, and it will be $-AB + AC = -A \times \overline{B-C}$.

This being sufficiently understood, the nine following propositions, and innumerable others of that kind, arising from the comparing of lines multiplied into themselves (which you may find done to your hand in *Vieta*, and other analytical Writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

Furthermore, * it appears that the product arising from the multiplication of any magnitude into the parts of any number is equal to the product arising from the multiplication of the same into the whole number: As $5A + 7A = 12A$, and $4A \times 5A + 4A \times 7A = 4A \times 12A$. Wherefore what is here delivered of the multiplying of right lines into themselves, the same may be understood of the multiplying of numbers into themselves, so that whatsoever is affirmed concerning lines in the nine following Theorems, holds good also in numbers; seeing they all immediately depend on, and are deriv'd from this first. * 19. ax.

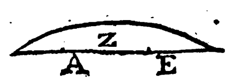
PROP. II.



If a right line Z *be divided any wise into two parts, the rectangles comprehended under the whole line* Z, *and each of the segments* A, E, *are equal to the square made of the whole line* Z.

I say that $ZA + ZE = Zq$. For take $B = Z$; *a* then is $BA + BE = BZ$, that is (because $B = Z$) $ZA + ZE = Zq$. *Which was to be demonstrated.* a 1. 2.

PROP. III.

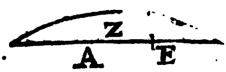


If a right line Z, *be divided any wise into two parts, the rectangle comprehended un-*

der the whole line Z, and one of the segments E, is equal to the rectangle made of the segments A, E, and the square described on the said segment E.

a 1. 2. I say ZE=AE+Eq. *a* For EZ=EA+Eq.

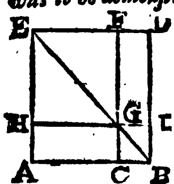
PROP. IV.



Or thus
$a+e=z$
$aa+2ae+ee=zz$

If a right line Z be cut any wise into two parts, the square described on the whole line Z, is equal to the squares described on the segments A, E, and to twice a rectangle made of the segments A, E taken together.

a 3. 2. I say that Zq=Aq+Eq+2AE. For ZA*a*=Aq+
b 2. 2. AE, and ZE*a* Eq+EA. Therefore whereas ZA+
c 1. ax. ZE *b* =Zq, *c* thence is Zq=Aq+Eq+2AE. *Which was to be demonstrated.*



Otherwise thus; Upon the right line AB make the square AD, and draw the diameter EB; thro' C, the point wherein the line AB is divided, draw the perpendicular CF; and thro' the point G draw HI parallel to AB.

d 4. cor. 32 1. Because the angle EHG = A is a right angle, and
e 32 1. AEB is *d* half a right, *e* therefore is the remaining
f 6. 1. angle HGE half a right angle. Therefore is HE *f* =
g 34. 1. HG *g* =EF *g* = AC, so that HF *h* is the square of the
h 29 def right line AC. After the same manner is CI proved to
k 19. ax. 1. be CBq. Therefore AG, GD, are rectangles under AC, CB, wherefore the whole square AD *k* =ACq + CBq + 2ACB. *Which was to be demonstrated.*

Coroll.

1. Hence it appears that the Pgrs which are about the diameter of a square are also squares themselves.
2. That the diameter of any square bisects its angles.
3. That if A = $\frac{1}{2}$ Z, then is Zq=4 Aq, and Aq = $\frac{1}{4}$ Zq. As on the contrary, if Zq=4 Aq, then is A= $\frac{1}{2}$ Z.

PROP. V.

A————|————|————B
C D

If a right line AB *be cut into equal parts* AC, CB, *and into unequal parts* AD, DB, *the rectangle comprehended under the unequal parts* AD, DB, *together with the square that is made of the difference of the parts* CD, *is equal to the square described on the half line* CB.

I say

I say that CBq = ADB + CDq.

For these are all equal; { CBq. / *a* CDq + CDB + DBq + CDB. / CDq + *b* CBD (*c* AC x BD) + CDB. / CDq + *d* ADB. }   a 4. 1. b 3. 2. c *hyp* d 1. 2.

This theorem is somewhat differently express'd and more easily demonstrated thus; *A Rectangle made of the sum and the difference of two right lines* A, E, *is equal to the difference of the squares of those lines.*

For if A + E be multiplied into A − E, * there ariseth Aq − AE + EA − Eq = Aq − Eq. *Which was to be demonstrated.*   * Sch 1. 2.

Schol.

If the line AB be divided otherwise, (*viz.*) nearer to the point of bisection, in E; then is AEB ⊏ ADB.

For AEB *a* = CBq − CEq, and ADB *a* = CBq − CDq. Therefore, whereas CDq ⊏ CEq, thence is AEB ⊏ ADB. *Which was to be demonstrated.*   a 5 2. & 3. *ax.*

Coroll.

1. Hence is ADq + DBq ⊏ AEq + EBq. For ADq + DBq + 2 ADB *b* = ABq *b* = AEq + EBq + 2 AEB. Therefore because 2AEB ⊏ 2ADB, shall ADq + DBq ⊏ AEq + EBq. *Which was to be demonstrated.*   b 4 2.

2. Hence is ADq + DBq − AEq *c* − EBq = 2 AEB − 2 ADB.   c 3. *ax.*

PROP. VI.

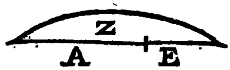
If a right line A *be divided into two equal parts, and another right line* E, *added to the same directly in one right line, then the rectangle comprehended under the whole and the line added,* (viz. A + E,) *and the line added* E, *together with the square which is made of* $\frac{1}{2}$ *the line* A, *is equal to the square of* $\frac{1}{2}$ A + E *taken as one line.*

I say that $\frac{1}{4}$ Aq (*a* Q. $\frac{1}{2}$ A) + AE + Eq = Q. $\frac{1}{2}$ A + E. *a* For, Q. $\frac{1}{2}$ A + E = $\frac{1}{4}$ Aq + Eq + AE. *Which was to be demonstrated.*   a 4 & 3. Cor. 4 2.

Coroll.

Hence it follows, that if 3 right lines E, E + $\frac{1}{2}$ A, E + A be in arithmetical proportion, then the rectangle contained under the extreme terms E, E + A, together with the square of the difference $\frac{1}{2}$ A, is equal to the square of the middle term E + $\frac{1}{2}$ A.

PROP. VII.



If a right line Z be divided any wise into two parts, the square of the whole line Z, together with the square made of one of the segments E, is equal to a double rectangle comprehended under the whole line Z, and the said segment E, together with the square made of the other segment A.

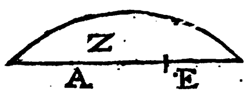
a 4. 2. I say that $Zq + Eq = 2ZE + Aq$. For Zq *a* $= Aq +$
b 3. 2. $Eq + 2\,AE$, and $2\,ZE$ *b* $= 2\,Eq + 2\,AE$. *Which was to be demonstrated.*

Coroll.

Hence it follows, that the square of the difference of any two lines Z, E, is equal to the squares of both the lines less by a double rectangle comprehended under the said lines.

c. 7. 2 *and* 3. *ax.* *c* For $Zq + Eq - 2ZE = Aq = Q\ Z - E$.

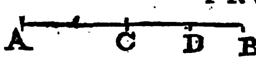
PROP. VIII.



If a right line Z be divided any wise into two parts, the rectangle comprehended under the whole line Z, and one of the segments E four times, together with the square of the other segment A, is equal to the square of the whole line Z, and the segment E, taken as one line Z+E.

a 7. 2. *and* 3. *ax.* I say that $4ZE + Aq = Q.\ Z + E$. For $2\,ZE$ *a* $= Zq$
$+ Eq - Aq$. Therefore $4\,ZE + Aq = Zq + Eq + 2ZE$ *b*
b 4. 2. $= Q\ Z + E$. *Which was to be demonstrated.*

PROP. IX.



If a right line AB be divided into equal parts AC, CB, and into unequal parts AD, DB, then are the squares of the unequal parts AD, DB, together, double to the square of the half line AC, and to the square of the difference CD.

I say that $ADq + DBq = 2\,ACq + 2\,CDq$. For $ADq +$
a 4. 2. DBq *a* $= ACq + CDq + 2\,ACD + DBq$. But $2\,ACD$
b *hyp.* (*b* $2BCD$) $+ DBq$ *c* $= CBq$ $(ACq) + CDq$. *d* Therefore
c 7. 2. $ADq + DBq = 2\,ACq + 2\,CDq$. *Which was to be de-*
d 2. *ax.* *monstrated.*

This may be otherwise delivered and more easily demonstrated thus; *the aggregate of the squares made of the sum and the difference of two right lines A, E, is equal to the double of the squares made from those lines.*

For

For Q: A+E *a*=Aq+Eq+2AE, and Q: A−E *b* =Aq+Eq −2AE. These added together make 2 Aq +2Eq. *Which was to be demonstrated.*

a 4. 2.
b *cor.* 7. 2.

PROP. X.

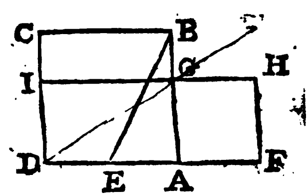
If a right line A *be divided into two equal parts, and another line be added in a right line with the same, then is the square of the whole line together with the added line (as being one line) together with the square of the added line* E, *double to the square of half* A, *and the added line* E, *taken as one line.*

I say that Eq+Q.A+E, *i. e. a* Aq+2Eq+2AE = 2Q. $\frac{1}{2}$A+2Q $\frac{1}{2}$A+E. For 2Q. $\frac{1}{2}$A *b* = $\frac{1}{2}$ Aq. And 2Q $\frac{1}{2}$A+E *c* = $\frac{1}{2}$ Aq+2Eq+2AE. *Which was to be demonstrated.*

a 4. 2.
b *cor.* 4. 2.
c 4. 2.

PROP. XI.

To cut a right line given AB, *in a point* G, *so that the rectangle comprehended under the whole line* AB, *and one of the segments* BG, *shall be equal to the square that is made of the other segments* AG.



Upon AB *a* describe the square AC. *b* Bisect the side AD in E, and draw the line EC; from the line EA produced take EF=EC. On AF make the square AH. Then is AH=AB x BG.

a 46. 1.
b 10. 1.

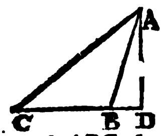
For HG being drawn out to I; the rectangle DH+EAq *c* =EFq *d* =EBq *e* =BAq+EAq: Therefore is DH *f* = BAq = to the square AC. Take away AI common to both, then remains the square AH=GC, that is, AGq=AB x BG. *Which was to be done.*

c 6. 2.
d *constr.*
e 47. 1.
f 3. *ax.*

Schol.

This proposition cannot be performed by numbers; * for there is no number that can be so divided, that the product of the whole into one part shall be equal to the square of the other part. * 6. 13.

PROP. XII.



In obtuse-angled triangles ABC, *the square that is made of the side* AC, *subtending the obtuse angle* ABC, *is greater than the squares of the sides* BC, AB, *that contain the obtuse angle* ABC, *by a double rectangle contained under one of the sides* BC, *which are about the obtuse angle* ABC, *on which side produced the perpendicular* AD *falls, and under the line* BD, *taken without the triangle from the point on which the perpendicular* AD *falls to the obtuse angle* ABC.

I say that ACq = CBq + ABq + 2CB x BD.

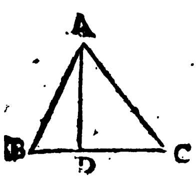
For these are all equal:
ACq.
a 47. 1. — a CDq + ADq.
b 4. 2. — b CBq + 2CBD + BDq + ADq.
c 47. 1. — CBq + 2 CBD + c ABq.

Scholium.

Hence, *the sides of any obtuse angled triangle* ABC *being known, the segment* BD *intercepted betwixt the perpendicular* AD, *and the obtuse angle* ABC, *as also the perpendicular it self* AD, *shall be easily found out.*

Thus, Let AC be 10, AB 7, CB 5. Then is ACq 100, ABq 49, CBq 25. And ABq + CBq = 74. Take that out of 100, then will 26 remain for 2 CBD. Wherefore CBD shall be 13; divide this by CB 5, there will $2\frac{3}{5}$ be found for BD. Whence AD will be found out by the 47. 1.

PROP. XIII.



In acute angled triangles ABC, *the square made of the side* AB, *subtending the acute angle* ACB, *is less than the squares made of the sides* AC, CB, *comprehending the acute angle* ACB, *by a double rectangle contained under one of the sides* BC, *which are about the acute angle* ACB, *on which the perpendicular* AD *falls, and under the line* DC, *taken within the triangle from the perpendicular* AD, *to the acute angle* ACB.

I

I say that $ACq + BCq = ABq + 2\,BCD$.

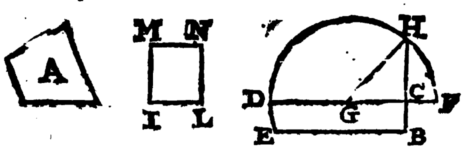
For these are equal. $\left\{\begin{array}{l} ACq + BCq. \\ a\ ADq + DCq + BCq. \\ b\ ADq + BDq + 2\,BCD. \\ c\ ABq + 2\,BCD. \end{array}\right.$ a 47. 1. b 7. 2. c 47. 1.

Coroll.

Hence, *the sides of an acute-angled triangle* ABC *being known, you may find out the segment* DC, *intercepted betwixt the perpendicular* AD, *and the acute angle* ABC, *as also the perpendicular it self* AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from ACq + BCq, that is, from $225 + 196 = 421$. Then remains 252 for 2 BCD, wherefore BCD will be 126, divide this by BC 14, then will 9 be found out for DC. From whence it follows, $AD = \sqrt{:225 - 81} = 12$.

PROP. XIV.



To find a square ML *equal to a right lined figure given* A.

a Make the rectangle DB = A, and produce the greater side thereof DC to F, so that CF = CB, *b* bisect DF, in G, about which as the center at the distance of GF describe the circle FHD, and draw out BC, till it meets the circumference in H. Then shall be CHq = *ML = A. a 45. 1. b 10. 1. * 46. 1.

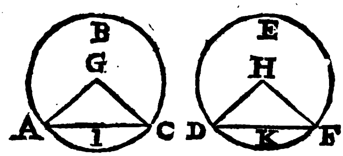
For let GH be drawn. Then is A *c* = DB *c* = DCF *d* = GFq − GCq *e* = HCq *c* = ML. *Which was to be done.* c *constr.* d 5. 2. *and* 3. *ax.* e 47. 1. *and* 3. *ax.*

The End of the second Book.

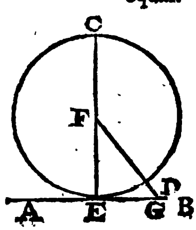
The

The Third Book. OF EUCLIDE's ELEMENTS.

Definitions.

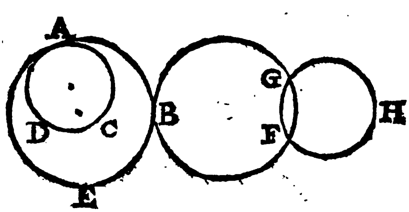


I. EQual circles (GABC, HDEF) are such whose diameters are equal; or, from whose centers right lines drawn GA, HD, are equal.



II. A right line *AB*, is said to touch a circle FEDC, when touching the same, and being produced, it cutteth it not.

The right line FG cuts the circle FEDC.

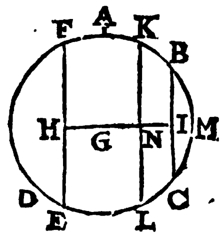


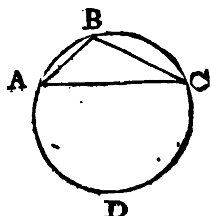
III. Circles DAC, ABE (and also FBG, ABE) are said to touch one the other, which touch, but cut not one the other

The

The circle BFG cuts the circle FGH.

IV. In a circle GABD, right lines FE, KL, are ſaid to be equally diſtant from the center, when perpendiculars GH, GN, drawn from the center G to them, are equal. And that line BC is ſaid to be furtheſt diſtant from it, on which the greater perpendicular GI falls.





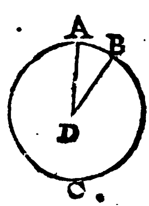
V. A ſegment of a circle (ABC) is a figure contained under a right line AC, and a portion of the circumference of a circle ABC.

VI. An angle of a ſegment CAB, is that angle which is contained under a right line CA, and the circumference of a circle AB.

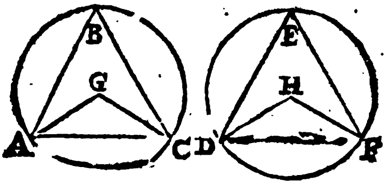
VII. An angle ABC is ſaid to be in a ſegment ABC, when in the circumference thereof ſome point B is taken, and from it right lines AB, CB, drawn to the ends of the right line AC, which is the baſe of the ſegment; then the angle ABC contained under the adjoined lines AB, CB, is ſaid to be an angle in a ſegment.

VIII. But when the right lines AB, BC, comprehending the angle ABC, do receive any periphery of the circle ADC, then the angle ABC is ſaid to ſtand upon that periphery.

IX. A ſector of a circle (ADB) is when an angle ADB is ſet at the center D of that circle; namely, that figure ADB comprehended under the right lines AD, BD, containing the angle, and the part of the circumference received by them AB.

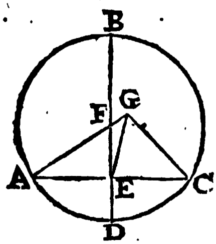


X. Like



X. Like ſegments of a circle (ABC, DFE) are thoſe which include equal angles (A B C, D E F;) or, in which the angles ABC, DEF, are equal.

PROP. I.



To find the center F *of a given circle* ABC.

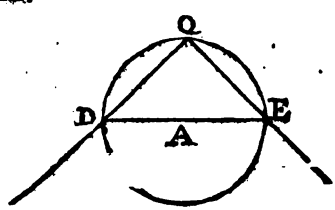
Draw a right line AC anywiſe in the circle, which biſect in E; thro' E draw a perpendicular DB, and biſect the ſame in F; the point F ſhall be the center.

If you deny it, let G, a point without the line BD, be the center (for it cannot be in the line BD, ſince that is divided unequally in every point but F;) let the lines G A, G C, GE, be drawn. Now if G be the center, *a* then is GA = GC, and AE=EC, by conſtruction, and the ſide GE common. *b* Therefore are the angles GEA, GEC, equal, and *c* conſequently right. *d* Therefore the angle GEC=FEG. *e Which is abſurd.*

a 15. *def.* 1.
b 8. 1
c 10. *def* 1.
d 12. *ax.*
e 9. *ax.*

Coroll.

Hence, if in a circle a right line BD biſect any right line AC at right angles, the center ſhall be in the cutting line BD.

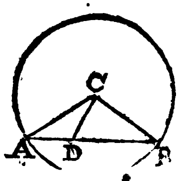


Andr. Tacq.

The center of a circle is eaſily found out by applying the top of a ſquare to the circumference thereof. For if the right line DE that joins the points D, E, in which the ſides of the

the square QD, QE, cut the circumference, be bisected in A, the point A shall be the center The demonstration whereof depends upon *Prop.* XXXI. of this Book.

PROP. II.



If in the circumference of a circle CAB, *any two points* A, B, *be taken, the right line* AB, *which joins those two points, shall fall within the circle*

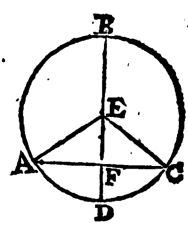
Take in the right line AB any point D; from the center C draw CA, CD, CB. Because CA *a* = CB, therefore is the angle A *b* = B. But the angle CDB *c* $\sqsupset$ A, therefore is CDB *d* $\sqsupset$ B, therefore CB *d* $\sqsupset$ CD But CB only reaches the circumference, therefore CD comes not so far; wherefore the point D is within the circle. The same may be proved of any other point in the line AB. And therefore the whole line AB falls within the circle *Which was to be dem.*

a 15. *def.* 1.
b 5. 1.
c 16. 1.
d 19. 1.

Coroll.

Hence, if a right line touch a circle, so that it cut it not, it touches but in one point.

PROP. III.



If in a circle EABC, *a right line* BD *drawn thro' the center, bisects any other line* AC, *not drawn thro' the center, it shall also cut it at right angles: And if it cuts it at right angles, it shall also bisect the same.*

From the center E let the lines EA, EC, be drawn.

1. *Hyp* Because AF *a* = FC, and EA *b* = EC, and the side EF common; the angles EFA, EFC, *c* shall be equal, and *d* consequently right. *Which was to be demonstrated*

2. *Hyp.* Because EFA *e* = EFC, and the angle EAF *f* = ECF, and the side EF common; *g* therefore is AF = FC. Therefore AC is cut into two equal parts. *Which was to be demonstrated.*

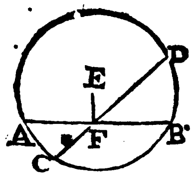
a *hyp.*
b 15 *def.* 1.
c 8 1.
d 10 *def* 1.
e *hyp* and 12. *ax.*
f 5. 1.
g 26. 1.

Coroll.

Coroll.

Hence, in any equilateral or Iſoſceles triangle, if a line drawn from the vertical angle biſect the baſe, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle biſects the baſe.

PROP. IV.

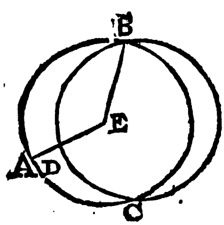


If in a circle ACD, *two right lines* AB, CD, *cut each other, and neither of them paſs thro' the center* E, *they ſhall not cut each other into equal parts.*

For if one line paſs thro' the center, 'tis plain it cannot be biſected by the other; becauſe by hypotheſis, the other does not paſs thro' the center.

If neither of them paſs thro' the center, then from the center E draw EF; now if *AB*, *CD*, were both
a 3. 3. biſected in F, then *a* would the angles EFB, EFD, be
b 9. *ax.* both right, and conſequently equal. b *Which is abſurd.*

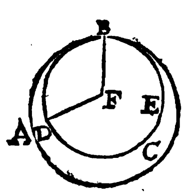
PROP. V.



If two circles BAC, BDC, cut each other, they ſhall not have the ſame center E.

For otherwiſe the lines *EB*, *EDA*, drawn from *E* the common center, would
a 15. *def.* 1 be DE *a* $=$ EB *a* $=$ EA, b
b 9. *ax.* *Which is abſurd.*

PROP. VI.

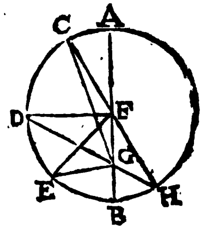


If two circles BAC, BDE, inwardly touch each other (in B) they have not one and the ſame center F.

For otherwiſe the right lines *FB*, *FDA*, drawn from the
a 15. *def.* 1. center *F*, would be *FD* *a* $=$
b 9 *ax.* *FB* *a* $=$ *FA*. b *Which is abſurd.*

PROP.

PROP. VII.



If in AB the diameter of a circle some point G be taken, which is not the center of the circle, and from that point certain right lines GC, GD, GE, fall on the circle the greatest line shall be that (GA) in which is the center F; the least, the remainder of the same line (GB.) And of all the other lines, the line GC, nearest to that which was drawn thro' the center is always greater than any line farther removed GD; and there can but two equal lines fall from the same point on the circle, viz. *one on each side of the least GB, or of the greatest GA.*

From the center *F* draw the right lines *FC, FD, FE*; * make the angle $BFH = BFE$. * 23. 1.

1. $GF + FC$ (that is GA) a $\sqsubset$ GC. *Which was to be demonstrated.* a 20. 1.

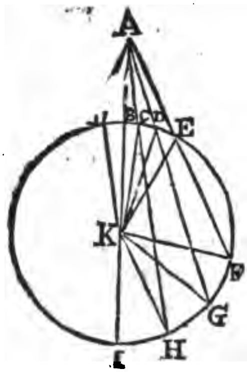
2. The side FG is common, and FC b $= FD$, and the angle GFC c $\sqsubset$ GFD; d wherefore the base $GC \sqsubset GD$. b 15 *def.* 1. c 9. *ax.* d 24. 1.

3. FB (FE) e $\sqsupset$ $GE + GF$. Therefore FG, which is common, being taken away from both, there remains $BG \sqsupset EG$. e 20. 1. f 5. *ax.*

4. The side FG is common, and $FE = FH$, and the angle BFH g $= BFE$; h Therefore is $GE = GH$. But that no other line GD from the point G, can be equal to GE, or GH, is already proved. *Which was to be demonstrated.* g *const.* h 4. 1.

PROP.

PROP. VIII.



If some point A be taken without a circle, and from that point be drawn certain right lines AI, AH, AG, AF, to the circle, and of those one AI, be drawn thro' the center K, and the others any wise; of all those lines that fall on the concave of the circumference, that is the greatest AI, which is drawn thro' the center; and of the others, that (AH) *which is nearest to the line that passes thro' the center, is greater than that which is more distant AG. But of all those lines that fall on the convex part of the circle, the least is that (AB) which is drawn from the point A, to the diameter IB; and of the others, that (AC) which is nearest to the least, is less than that which is farther distant AD. And from that point there can be only two equal right lines AC, AL, drawn, which shall fall on the circumference on each side of the least line AB, or of the greatest AI.*

From the center K, draw the right lines KH, KG, KF, KC, KD, KE, and make the angle $AKL = AKC$.

a 20. 1. 1. AI $(AK + KH)$ *a* $\sqsubset AH$.

b 24. 1 2. The side AK is common, and $KH = KG$, and the angle $AKH \sqsubset AKG$; *b* therefore the base $AH \sqsubset AG$.

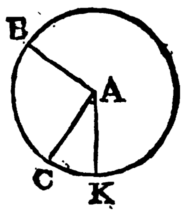
c 20. 1. 3. KA *c* $\sqsupset KC + CA$. From hence take away KC,
d 5. ax. KB, which are equal; then will remain AB, *d* $\sqsupset AC$.
e 21 1. 4. $AC + CK$ *e* $\sqsupset AD + DK$. Take from both, CK,
f 5. ax. DK, which are equal; then remains AC *f* $\sqsupset AD$.

5. The side KA is common, and $KL = KC$, and the
g constr. angle AKL *g* $= AKC$; *h* therefore $LA = CA$. But that
h 4. 1. no other line could be drawn equal to these, was proved above. Therefore, &c.

PROP

PROP. IX.

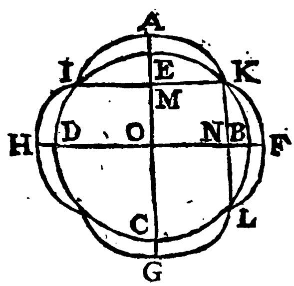
If in a circle B C K, *a point* A *be taken, and from that point more than two equal right lines* A B, A C, A K, *can be drawn to the circumference, then is that point* A *the center of the Circle.*



For *a* from no point without the center can more than two equal right lines be drawn to the circumference Therefore A is the center. *Which was to be demonstrated.* a 7. 3.

PROP. X.

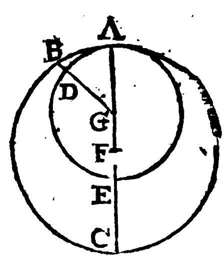
A circle IAKBL, *cannot cut another circle* IEKFL, *in more than two points.*



Let one circle, if it may be, cut the other in three points I, K, L, and IK, KL, being join'd, let them be bisected in M and N. *a* Both circles have their centers in their perpendiculars MC, NH, and in the intersection of those perpendiculars which is O. Therefore the circles that cut each other have the same center. Which is false, by *Prop.* 5. 3. a *cor.* 1. 3.

PROP. XI.

If two circles G A D E, F A B C, *touch one the other inwardly, and their centers be taken* G, F; *a right line* FG *joining their centers, and produced, shall cut the circumference in* A, *the point of contact of the circles.*

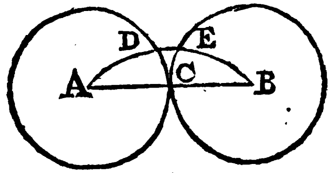


If it can be, let the right line F G produced cut the circles in some other

D point

point than A; so that not FGA, but FGDB, shall be a right line. Let the line GA be drawn. Now, because
a 15 *def.* 1. GD $a=$ GA, and GB $b\sqsupset$ GA (since the right line FGB
b 7, 3. passes thro' F, the center of the greater circle) there-
c 9. *ax.* fore is GB $\sqsupset$ GD. c *Which is absurd.*

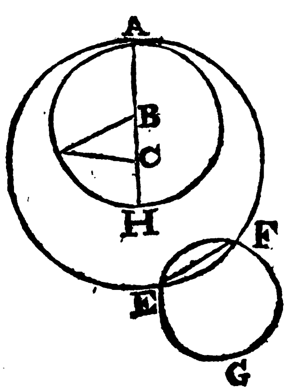
PROP. XII.



If two circles ACD, BCE. *touch one the other outwardly, the right line* AB, *which joins their centers* A, B, *shall pass thro' the point of contact* C.

If it may be, let ADEB be a right line cutting the circles, not in the point of contact C, but in the points
a 20. 1. D, E; draw AC, CB, then is AD + EB (AC + CB) *a*
b 9. *ax.* $\sqsubset$ ADEB. b *Which is absurd.*

PROP. XIII.



A circle CAF *cannot touch a circle* BAH *in more points than one* A, *whether it be inwardly or outwardly.*

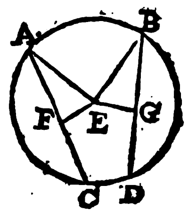
1. Let one circle (if it can be) touch another in two points A,
a 11. 3. H. *a* Then will the right line CB, that joins the centers, if it be produced, fall as well in A as H. Now because CH $b=$ CA, and
b 15. *def.* 1. BH $\sqsubset$ CH, therefore
c 15 *def.* 1. is BA (c BH) $\sqsubset$ CA.
d 9. *ax.* d *Which is absurd.*

2. If

2. If it be said to touch outwardly in the points E and F, then draw the line EF, *e* which will be in both circles. Therefore those circles cut one the other; which is against the Hyp. e 2. 3.

PROP. XIV.

In a circle EABC, equal right lines AC, BD, are equally distant from the center E: and right lines AC, BD, which are equally distant from the center, are equal among themselves.



From the center *E*, draw the perpendiculars *EF*, *EG*, *a* which will bisect the lines *AC*, *BD*, join *EA*, *EB*. a 3. 3.

1. *Hyp.* $AC = BD$, therefore AF *b* $= BG$. But also $EA = EB$; therefore FEq *c* $= EAq - AFq = EBq - BGq$ *c* $= EGq$. *d* Therefore $FE = EG$. b 7. *ax.* c 47. 1. *and* 3. *ax.*

2. *Hyp.* $EF = EG$. Therefore AFq *e* $= EAq - EFq$ $= EBq - EGq = BGq$. Therefore AF *d* $= GB$, and *e* consequently $AC = BD$. *Which was to be demonstrated.* d *schol.* 48. 1. e 6. *ax.*

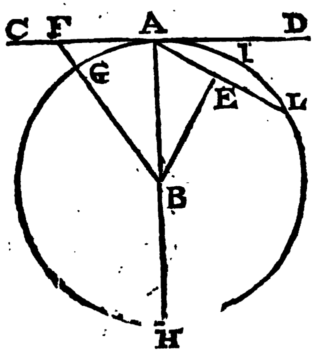
PROP. XV.

In a circle GABC, the greatest line is the diameter AD; and of all other lines, that FE, which is nearest to the center G, is greater than any line BC farther distant from it.

1 Draw *GB*, and *GC*. The diameter *AD* (*a* $GB + GC$) *b* $\sqsubset BC$. a 15. *def.* 1. b 20. 1.

2. Let the distance *GI* be $\sqsubset GH$. Take $GN = GH$ Thro' the point *N* draw *KL* perpendicular to *GI*: join *GK*, *GL*. Because $GK = GB$, and $GL = GC$. and the angle $KGL \sqsubset BGC$; *c* therefore is *KL* (*FE*) $\sqsubset BC$. *Which was to be demonstrated.* c 24. 1.

PROP. XVI.



A line CD, *drawn from the extreme point of the diameter* HA, *of a circle* BALH, *perpendicular to the said diameter, shall fall without the circle; and between the same right line and the circumference, cannot be drawn another line* AL. *And the angle of the semicircle* BAI, *is greater than any right-lined acute angle* BAL; *and the remaining angle without the circumference* DAI, *is less than any right-lined angle.*

1. From the center B, to any point F, in the right line AC, draw the right line BF. The side BF subtending the right angle BAF, is *a* greater than the side BA, which is opposite to the acute angle BFA. Therefore, whereas BA, (BG) reaches to the circumference, BF shall reach further; and so the point F, and for the same reason any other point of the line AC, shall be without the circle.

a 19. 1.

2. Draw BE perpendicular to AL. The side BA, opposite to the right angle BEA, is *b* greater than the side BE, which subtends the acute angle BAE; therefore the point E, and so the whole line EA, falls within the circle.

b. 19. 1.

3 Hence it follows, that any acute angle, to wit, EAD, is greater than the angle of contact DAI, and that any acute angle BAL is less than the angle of a semicircle BAI. *Which was to be dem.*

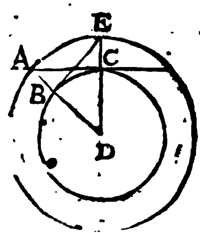
Coroll.

• Hence, a right line drawn from the extremity of the diameter of a circle, and at right-angles, is a tangent to the said circle.

From this proposition are gathered many paradoxes, and wonderful consectaries, which you may meet with in the interpreters.

PROP.

PROP. XVII.



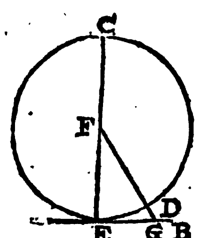
From a point given A, *to draw a right line* AC, *which shall touch a circle given* DBC.

From D, the center of the circle given, to the given point A, let the line DA be drawn, cutting the circumference in B, from the center D, describe another circle thro' the point A; and from B, draw a perpendicular to AD, which shall meet with the circle AE in the point E; and draw ED meeting with the circle BC, in the point C. Then a line drawn from A to C, shall touch the circle DBC.

For DB *a* = DC, and DE *a* = DA, and the angle D is common; *b* therefore the angle ACD = EBD and right. *c* Therefore AC, touches the circle in C. *Which was to be done.*

a 15. *def.* 1.
b 4. 1.
c *cor.* 16 3.

PROP. XVIII.

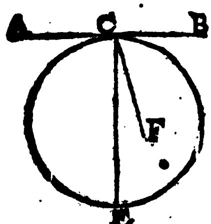


If any right line AB *touches a circle* FEDC, *and from the center to the point of contact* E, *a right line* FE *be drawn; that line* FE *shall be perpendicular to the tangent* AB.

If you deny it, let some other line FG be drawn from the center F, perpendicular to the tangent, and *a* cutting the circle in D. Therefore, whereas the angle FGE is said to be right, *b* thence is the angle FEG acute; *c* so that FE (FD) ⊏ FG. *d Which is absurd.*

a *def.* 2. 3.
b *cor.* 17. 1.
c 19. 1.
d 9. *ax.*

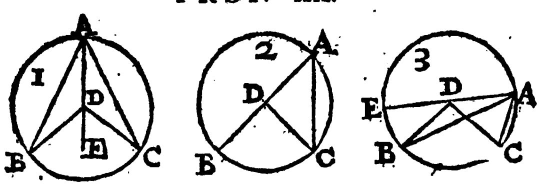
PROP. XIX.



If any right line AB *touch a circle, and from the point of contact* C, *a right line* CE *be erected at right angles to the tangent, the center of the circle shall be in the line* CE *so erected*

If you deny it, let the center be without the line CE, in the point F; and from F, to the point of contact, let FC be drawn. Therefore the angle FCB is right,
a 12. *ax.* and *a* consequently equal to the angle ECB, which was
b 9. *ax.* right by Hypothesis. b *Which is absurd.*

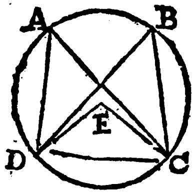
PROP. XX.



In a circle DABC, *the angle* BDC *at the center is double of the angle* BAC *at the circumference, when the same arch of the circle* BC, *is the base of the angles.*

Draw the Diameter ADE. The outward angle BDE
a 32. 1. *a*= DAB+ DBA *b*=2DAB : In like manner the an-
b 5. 1. gle EDC =2DAC, Therefore in the first case *c* the
c 2 *ax.* whole angle BDC = 2 BAC, and in the third case the
d 20 *ax.* remaining angle BDC *d* = 2 BAC. *Which was to be demonstrated.*

PROP. XXI.

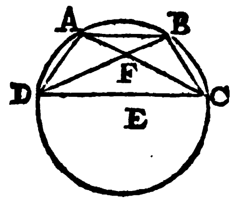


In a circle EDAC, *the angles* DAC, *and* DBC, *which are in the same segment, are equal one to the other*

1 *Case.* If the segment DABC be greater than a semicircle, from the center E draw ED, EC. Then is
a 20. 3. twice the angle A *a*=E *a* = 2 B. *Which was to be demonstrated.*

2. *Case*

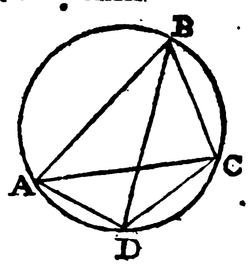
2. *Case.* If the segment be less than a semicircle, then is the sum of the angles of the triangle ADF equal to the sum of the angles of the triangle BCF, from each let AFD equal to BFC, *b* and ADB *c* = ACB, be taken away, then remains DAC=DBC. *Which was to be demonstrated.*



b 15 1.
c *by the* 1st *case.*

PROP. XXII.

The angles ADC, ABC, *of a quadrilateral figure* ABCD, *described in a circle, which are opposite one to the other, are equal to two right angles.*



Draw AC, BD. The angle ABC + BCA + BAC *a* = 2 right. But BDA *b* = BCA, and BDC *b* = BAC. *c* Therefore ABC + ADC = 2 right angles. *Which was to be demonstrated.*

a 32. 1.
b 21 3.
c 1. *ax.*

Coroll.

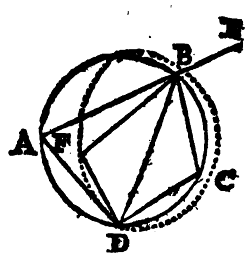
1. Hence, if one side * AB of a quadrilateral, described in a circle, be produced, the external angle EBC is equal to the internal angle ADC, which is opposite to that ABC, which is adjacent to EBC, as appears by 13. 1. and 3. *ax.*

* *See the following Diagr.*

2. A circle cannot be described about a Rhombus because its opposite angles are greater, or less than two right angles.

Schol.

If in a quadrilateral ABCD, *the angles* A, *and* C, *which are opposite, be equal to two right, then a circle may be described about that quadrilateral.*

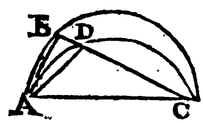


For a circle will pass through any three angles B, C, D, (as shall appear by 5. 4) I say that it shall also pass thro' A the 4th angle of such a quadrilateral. For if you deny it, let the circle pass thro' F: Therefore

fore the right lines BF, FD, BD being drawn, the angle C+F a= 2 right b = C+A; wherefore A c is equal to F. *d Which is absurd.*

a 22. 3.
b *hyp.*
c 3. *ax.*
d 21. 1.

PROP. XXIII.

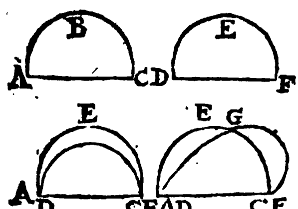


Two like and unequal segments of circles ABC, ADC, *cannot be set on the same right line* AC, *and on the same side thereof.*

For if they are said to be like, draw the line CB cutting the circumferences in D and B, join AB and AD. Because the segments are supposed like, *a* therefore is the angle ADC=ABC. b *Which is absurd.*

a 10. *def.* 3
b 16. 1.

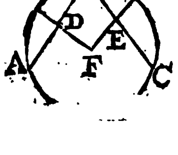
PROP. XXIV.



Like segments, of circles ABC, DEF *upon equal right lines* AC, DF, *are equal one to the other.* The base AC being laid on the base DF, will agree with it, because AC=DF. Therefore the segment ABC shall agree with the segment DEF (for otherwise it shall fall either within or without; and if so *a* then the segments are not like, which is contrary to the Hypothesis, and at least it shall fall partly within and partly without, and so cut in three points, *b* which is absurd. *c* Therefore the segment ABC=DEF. *Which was to be demonstrated.*

a 23. 3.
b 10. 3.
c 8. *ax.*

PROP. XXV.



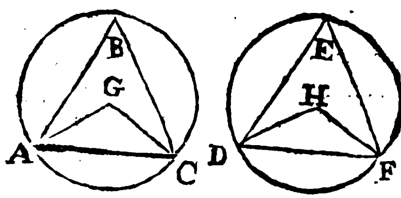
A segment of a circle ABC, *being given, to discribe the whole circle whereof that is a segment.*

Let two right lines be drawn AB, BC, which bisect in the points D and E. From D and E draw the perpendiculars DF, EF, meet-

meeting in the point F. I say this point shall be the center of the circle.

For the center shall be as well in *a* DF as EF, therefore it must be in the point F, which is common to them both. *Which was to be done.* a *cor.* 1. 3

PROP. XXVI.



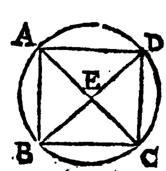
In equal circles GABC, HDEF, *equal angles stand upon equal parts of the circumference*, AC, DF ; *whether those angles be made at the centers* G, H, *or at the circumferences*, B, E.

Because the circles are equal, therefore is GA=HD, and GC = HF ; also by Hypothesis the angle G=H ; *a* therefore AC=DF. Moreover the angle B*b*=½G*c*= ½ H*b*=E. *d* Therefore the segments ABC, DEF are like, and *e* consequently equal, *f* whence the remaining segments alse AC, DF, are equal. *Which was to be dem.*

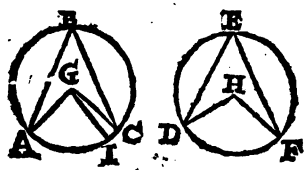
a 4. 1.
b 20 3.
c *hyp.*
d 10 *def* 3
e 24 3.
f 3. *ax.*

Schol.

In a circle ABCD, let an arch AB be equal to DC; then shall AD be parallel to BC. For the right line AC being drawn, the angle ACB *a* = CAD; wherefore by 27. 1. the said sides are parallel. a 26 3.



PROP. XXVII.

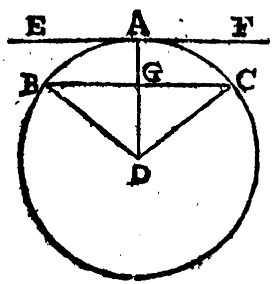


In equal circles GABC, HDEF, *the angles standing upon equal parts of the circumference.* AC, DF, *are equal between themselves, whether they be made at the centers* G, H, *or at the circumferences,* B, E.

For if it be possible, let one of the angles AGC be $\sqsubset$ DHF, and make AGI = DHF; thence is the arch AI *a* = DF *b* = AC. c *Which is absurd.*

a 26. 3.
b *hyp.*
c 9. *ax.*

Schol.



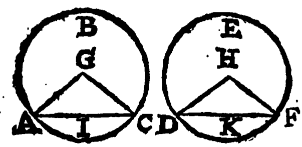
A right line EF, *which, being drawn from* A *the middle point of any periphery* BC, *toucheth the circle, is parallel to the right line* BC, *subtending the said periphery.*

From the center D draw a right line DA to the point of contact A, and join DB, DC.

The side DG is common, and DB = DC, and the angle BDA *a* = CDA, (because the arches BA, CA are *b* equal) therefore the angles at the base DGB, DGC are *c* equal, and *d* consequently right; but the inward angles GAE, GAF are also *e* right, *f* therefore BC, EF are parallel. *Which was to be demonstrated.*

a 27. 3.
b *hyp.*
c 4. 1.
d 10. *def.* 1.
e *hyp.*
f 28. 1.

PROP. XXVIII.



In equal circles GABC, HDEF, *equal right lines* AC, DF, *cut off equal parts of the circumference, the greatest* ABC, *equal to the greatest* DEF, *and the least* AIC *to the least* DKF.

From

From the centers G, H, draw GA, GC, and HD, HF.

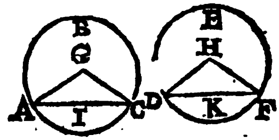
Becauſe GA=HD, and GC=HF, and AC *a* = DF, *b* therefore is the angle G=H; *c* whence the arch AIC =DKF; *d* and ſo the remaining arch ABC=DEF. *Which was to be demonſtrated.*

a *hyp.*
b 8. 1.
c 26. 3.
d 3. *ax.*

But if the ſubtended line AC be $\sqsubset$ or $\sqsupset$ than DF, then in like manner will the arch AC be $\sqsubset$ or $\sqsupset$ than DF.

PROP. XXIX.

In equal circles GABC, HDEF, *the right lines* AC, DF, *which ſubtend equal peripheries* ABC, DEF, *are equal*



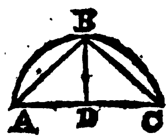
Draw the lines GA, GC, and HD, HF. Becauſe GA=HD, and GC=HF, and (becauſe the arches AC, DF are *a* equal) the angle G *b*=H, *c* therefore is the baſe AC=DF. *Which was to be demonſtrated.*

a *hyp.*
b 27. 3.
c 4. 1.

This and the three precedent propoſitions may be underſtood alſo of the ſame circle.

PROP XXX.

To cut a Periphery given ABC *into two equal parts.*



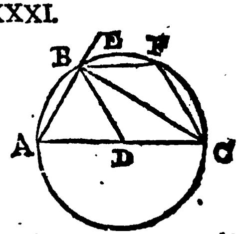
Draw the right line AC, and biſect it in D; from D draw a perpendicular DB meeting with the arch in B, it ſhall biſect the ſame

For join AB, and CB. The ſide DB is common, and AD *a*=DC, and the angle ADB *b*=CDB. *c* Therefore AB=BC; *d* whence the arch AB=BC *Which was to be done.*

a *conſtr.*
b 12. *ax.*
c 4. 1.
d 28. 3.

PROP. XXXI.

In a circle the angle ABC, *which is in the ſemicircle, is a right angle; but the angle, which is in the greater ſegment* BAC, *is leſs than a right angle, and the angle which is in the leſſer ſegment* BFC *is greater than a right angle. Moreover, the angle of the greater ſegment is greater than a right angle, and the angle of the leſſer ſegment is leſs than a right angle.*



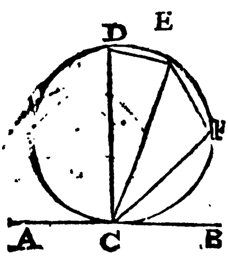
From

From the center D draw DB. Because DB = DA,
a 5. 1. therefore is the angle A *a* = DBA, and the angle DCB
b 2. ax. *a* = DBC, *b* therefore the angle ABC = A + ACB,
c 32 1. *c* = EBC, *d* so that ABC and EBC are right angles.
d 10 def 1. *W: W. to be dem.* *e* Therefore BAC is an acute angle.
e cor. 17. 1. *W. W to be dem.* And further, whereas BAC + BFC *f*
f 22. 3. = 2 right, therefore BFC is an obtuse angle. Lastly, the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle made by the right line CB, and the periphery of the lesser segment BFC *g* is less than the right
g 9. ax. angle EBC. *Which was to be demonstrated.*

Schol.

In a right angled triangle ABC, *if the hypothenuse (or line subtending the right angle)* AC *be bisected in* D, *a circle drawn from the center* D *through the point* A *shall also pass through the point* B; *as you may easily demonstrate from this* prop and 21. 1.

+ PROP. XXXII.

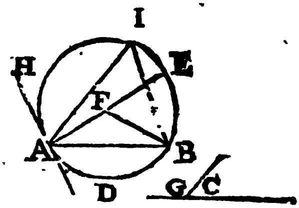


If a right line AB *touch a circle, and from the point of contact be drawn a right line* CE, *cutting the circle, the angles* ECB, ECA, *which it makes with the tangent line, are equal to those angles* EDC, EFC, *which are made in the alternate segments of the circle*

Let CD, the side of the
a 26. 3. angle EDC be perpendicular to AB (*a* for it's to the
b 19. 3. same purpose) *b* therefore CD is the diameter, *c* there-
c 31. 3. fore the angle CED in a semicircle is a right angle,
d 32. 1. *d* and therefore the angle D + DCE = to a right angle *e*
e constr. = ECB + DCE. *f* Therefore the angle D = ECB. *Which*
f 3. ax. *was to be dem.*
g 13. 1. Now whereas the angle ECB + ECA *g* = 2 right
h 22 3. *h* = D + F, from both of these take away ECB and
k 3. ax. D, which are equal, *k* then remains ECA = F. *Which was to be dem.*

PROP. XXXIII.

Upon a right line AB *to describe a segment of a circle* AIEB *which shall contain an angle* AIB, *equal to a right lined angle given* C.

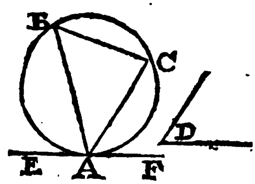


a Make the angle BAD=C. Through the point A draw the line AE perpendicular to HD. At the other end of the line given AB make an angle ABF=BAF, one of the sides of which shall cut the line AE in F; from the center F through the point A, describe a circle, which shall pass through B. (Because the angle FBA *b*=FAB, and *c* therefore FB=FA.) AIB is the segment sought. For because HD is perpendicular to the diameter AE, therefore HD *d* touches the circle which AB cuts. And therefore the angle AIB *e*= BAD *f*=C. *Which was to be done.*

a 23. 1.
b *constr.*
c 6. 1.
d *cor.* 16. 3.
e 32. 3.
f *constr.*

PROP. XXXIV.

From a circle given ABC *to cut off a segment* ABC *containing an angle* B *equal to a right lined angle given* D

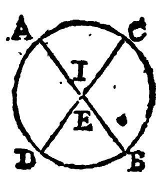


a Draw a right line EF which shall touch the circle given in A, *b* let AC be drawn also making an angle FAC=D. This line shall cut off ABC containing an angle B *c*=CAF *d*=D. *Which was to be done.*

a 17. 3.
b 23. 1.
c 32. 3.
d *constr.*

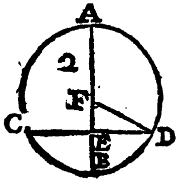
PROP. XXXV.

If in a circle DBCA *two right lines* AB, DC *cut each other, the rectangle comprehended under the segments* AE, EB, *of the one, shall be equal to the rectangle comprehended under the segments* CE, ED *of the other.*



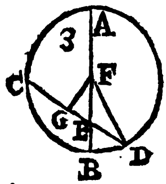
1. *Case.* If the right lines cut one the other in the center, the thing is evident.

2. *Case*



a 5. 2.
b *sch*48. 1.
c 47. 1.
d *hyp.*
e 3. *ax.*

2. *Case* If one line AB passes thro' the center F, and bisects the other line CD, then draw FD. Now the rectangle AEB + FEq *a* = FBq. *b* = FDq. *c* = EDq + FEq *d* = CED + FEq. *e* Therefore the rectangle AEB = CED. *Which was to be demonstrated.*

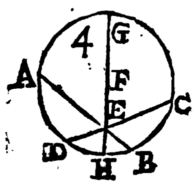


3. *Case.* If one of the lines AB be the diameter, and cut the other line CD unequally, bisect CD by FG a perpendicular from the center.

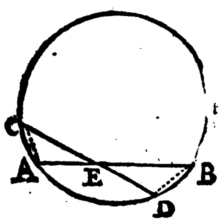
f 5. 2.
g 47. 1.
h 5. 2.
k 47. 1.
l 3. *ax.*

These are equal: The rectangle AEB + FEq. *f* FBq (FDq) *g* FGq + GDq FGq + *h* GEq + Rectang. CED. *k* FEq + CED.

l Therefore the rectangle AEB = CED.



4. *Case.* If neither of the right lines AB, CD pass thro' the center, then through the point of intersection E, draw the diameter GH. By that which hath been already demonstrated, it appears that the rectangle AEB = GEH = CED. *Which was to be demonstrated*



a 15. 1.
b 21. 3
c *cor.* 32. 1.
d 4. 6.
e 16. 6.

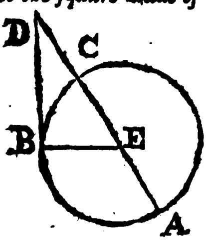
More easily, and generally, thus; join AC and BD, then because the angles *a* CEA, DEB, and *b* also C, B (upon the same arch AD) are equal. thence are the triangles CEA, BED, *c* equiangular. *d* Wherefore CE, EA : : EB, ED, and *e* consequently CE x ED = AE x EB. *Which was to be demonstrated.*

The

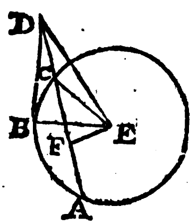
The citations out of the 6. Book, both here and in the following *prop.* have no dependance on the same; so that it was free to use them.

PROP. XXXVI.

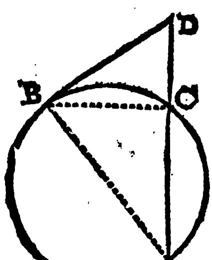
If any point be taken without a circle EBC, *and from that point two right lines* DA, DB, *fall upon the circle, whereof one* DA *cuts the circle, the other* DB *touches it, the rectangle comprehended under the whole line* DA *that cuts the circle, and* DC, *that part which is taken from the point given* D *to the convex of the periphery, shall be equal to the square made of the tangent line.*



1. *Case* If the secant AD passes thro' the center, then join EB,
this *a* will make a right angle a 18. 3.
with the the line DB, wherefore
DBq + *d* EBq (ECq) *b* = EDq b 47. 1.
c = AD x DC + ECq. There- c 6. 2.
fore AD x DC = DBq. *Which* d 3. *ax.*
was to be demonstrated.



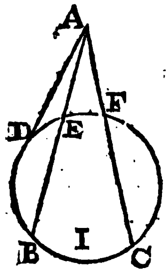
2. *Case.* But if AD passes not thro' the center, then draw EC, EB, ED, and EF perpendicular
to AD, *a* wherefore AC is bi- a 3. 3.
sected in F.
Because BDq + EBq *b* = DEq b 47. 1.
b = EFq + FDq *c* = EFq + ADC c 6. 2.
+ FCq *d* = ADC + CEq (EBq) d 47. 1.
e Therefore is BDq = ADC. e 3 *ax.*
Which was to be demonstrated.



More easily, and generally thus; draw AB and BC. Then,
because the angles A, and D-
BC *a* are equal, and the angle a 32 3.
D common to both, thence
are the triangles BDC, ADB *b* b 32. 1.
equiangular *c* Wherefore AD, c 4. 6.
DB : : DB, CD, and *d* conse- d 17. 6.
quently AD x DC = DBq.
Which was to be demonstrated.

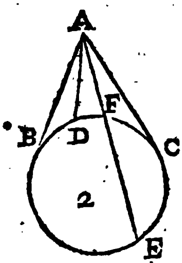
Coroll.

Coroll.



1. Hence, If from any point A, taken without a circle, there be several lines AB, AC drawn which cut the circle; the rectangles comprehended under the whole lines AB, AC, and the outward parts AE, AF, are equal between themselves.

For if the tangent AD be drawn,
a 36. 3. then is CAF = ADq *a* = BAE.



2. It appears also from hence, that if two lines AB, AC, drawn from the same point do touch a circle, those two lines are equal one to the other.

For if AE be drawn cutting the
a 36. 3. circle, then is ABq *a* = EAF *b* =
b 36 3. ACq.

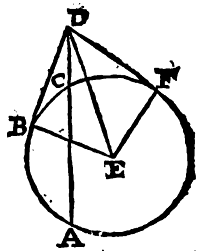
3. It is also evident, that from a point A taken without a circle, there can be drawn but two lines AB, AC that shall touch the circle.

For if a third line AD be said to touch the circle,
c 2. *cor.* thence is AD *c* = AB *c* = AC. d *Which is absurd.*
d 8. 3. 4. And on the contrary, it is plain, that if two equal right lines AB, AC, fall from any point A, upon the convex periphery of a circle, and that if one of these equal lines AB touch the circle, then the other AC touches the circle also.

For if possible, let not AC, but another line AD,
e 2. *cor.* touch the circle; therefore is AD *e* = AC *f* = AB.
f *hyp.* g *Which is absurd.*
g 8 3.

PROP.

PROP. XXXVII.



If without a circle EBF *any point* D *be taken, and from that point two right lines* DA, DB *fall on the circle, whereof one line* DA *cuts the circle, the other* DB *falls upon it; and if also the rectangle comprehended under the whole line that cuts the circle, and under that part of it* DC *which is taken betwixt the point* D *and the convex periphery, be equal to that square which is made of the line* DB *falling on the circle, I say that that line* DB *so falling shall touch the circle given.*

From the point D *a* let a tangent DF be drawn, and from the center E draw ED, EB, EF. Now because DBq *b* = ADC *c* = DFq, therefore is DB *d* = DF: But EB = EF, and the side ED common; *e* therefore the angle EBD = EFD; but EFD is a right angle, and *f* therefore EBD is right also; and *g* therefore DB touches the circle. *Which was to be dem.*

a 17. 3.
b *hyp.*
c 36. 3.
d 1. *ax.* &
sch. 4. 1.
e 8. 1.
f 12. *ax.*
g *cor.* 16. 3.

Coroll.

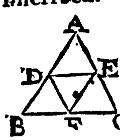
From hence it follows that the *h* angle EDB = EDF. h 8. 1.

The End of the third Book.

The FOURTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

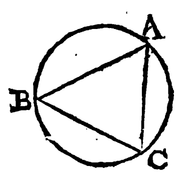
I. A Right-lined figure is ſaid to be inſcribed in a right-lined figure, when every one of the angles of the inſcribed figure touch every one of the ſides of the figure wherein it is inſcribed.

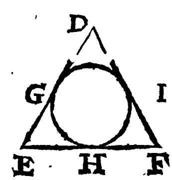


So the triangle DEF *is inſcribed in the triangle* ABC.

II. In like manner a figure is ſaid to be deſcribed about a figure, when every one of the ſides of the figure circumſcribed touch every one of the angles of the figure about which it is circumſcribed.

So the triangle ABC *is deſcribed about the triangle* DEF.





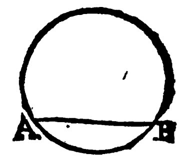
III. A right-lined figure is ſaid to be inſcribed in a circle, when all the angles of that figure which is inſcribed do touch the circumference of the circle.

IV A right-lined figure is ſaid to be deſcribed about a circle, when all the ſides of the figure which is circumſcribed touch the periphery of the circle

V. After the like manner a circle is ſaid to be inſcribed in a right-lined figure, when the periphery of the circle touches all the ſides of the figure, in which it is inſcribed.

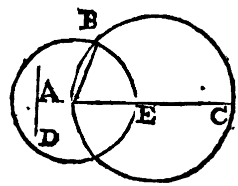
VI. A circle is said to be described about a figure when the periphery of the circle touches all the angles of the figure, which it circumscribes.

VII. A right line is said to be fitted or applied in a circle when the extremes thereof fall upon the circumference; *as the right line* AB.



PROP. I.

In a circle given ABC *to apply a right line* AB *equal to a right line given* D, *which doth not exceed* AC *the diameter of the circle.*

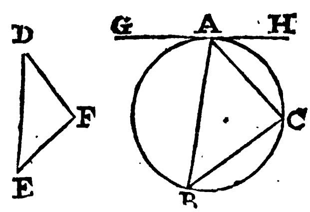


From the center A at the distance AE = D *a* describe a circle meeting with the circle given in B, draw AB. Then is AB *b* = AE *c* = D. *Which was to be done.*

a 3. *post.* & 3. 1.
b 15 *def.* 1.
c *constr.*

PROP. II.

In a circle given ABC *to describe a triangle* ABC, *equiangular to a triangle given* DEF.

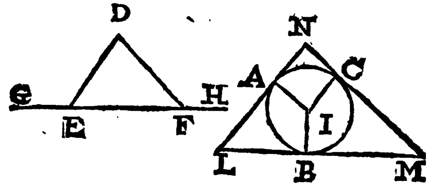


Let the right line GH *a* touch the circle given in A; *b* make the angle HAC = E, *b* and the angle GAB = F, then join BC; and the thing is done.

For the angle B *c* = HAC *d* = E, and the angle C *c* = GAB *d* = F; *e* whence also the angle BAC = D. Therefore the triangle BAC inscribed in the circle is equiangular to DEF. *Which was to be done.*

a 17. 3.
b 23. 1.
c 32. 3.
d *constr.*
e 32. 1.

PROP. III.



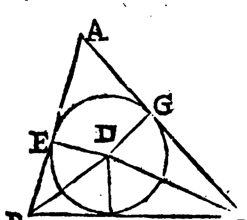
About a circle given IABC *to describe a triangle* LNM *equiangular to a triangle given* DEF.

a 23. 1. Produce the side EF on both sides; at the center I *a* make an angle AIB=DEG, and an angle BIC=DFH. Then in the points A, B, C, let three right lines LN,
b 17. 3. LM, NM, *b* touch the circle, and the thing is done.

For it's evident that the right lines LN, LM, MN,
c 13. *ax.* will meet and make a triangle, *c* because the angles
d 11. 3. LAI, LBI are right; so that if the *d* right line AB was drawn it would make the angles LAB, LBA, less than two right angles.

e *sch.* 32. 1. Since therefore the angle AIB+L *e*= 2 right angles
f 13. *ax.* *f* = DEG+DEF, and AIB *g* =DEG; *h* therefore
g *constr.* is the angle L=DEF By the like way of argument
h 3. *ax.* the angle M = DFE. *k* Therefore also the angle
k 32. 1. N=D. And therefore the triangle LNM described about the circle is equiangular to EDF the triangle given. *Which was to be done.*

PROP. IV.



In a triangle given ABC, *to inscribe a circle* EFG.

a 9. 1. *a* Bisect the angles B and C with the right lines
b 12. 1. BD, CD, meeting in the point D, *b* and draw the perpendiculars DE, DF, DG. A circle described from the center D thro' E, will pass through G and F, and touch the three sides of the triangle

c *constr.* For the angle DBE *c* = DBF; and the angle DEB *d*
d 12. *ax.* = DFB; and the side DB common, *e* therefore DE=
e 26. 1. DF.

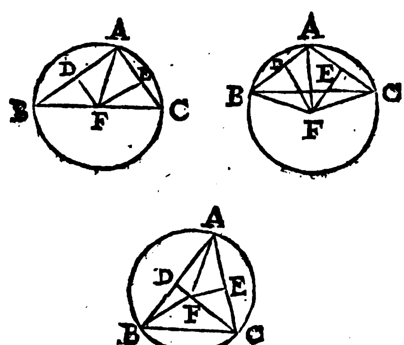
DF. By the like argument DG = DF, The circle therefore described from the center D passes through the three points E, F, G, and whereas the angles at E, F, G, are right, therefore it touches all the sides of the triangle. *Which was to be done.*

Schol.

Hence, *The sides of a triangle being known, their segments which are made by the touchings of the circle inscribed, shall be found*, Thus; *Pet. Herig*

Let AB be 12, AC 18, BC 16, then is AB + BC = 28. Out of which subduct 18 = AC = AE + FC, then remains 10 = BE + BF. Therefore BE, or BF = 5; and consequently FC, or CG = 11. Wherefore GA, or AE, = 7.

PROP. V.



About a triangle given ABC, *to describe a circle* FABC.

a Bisect any two sides BA, CA with perpendiculars DF, EF, meeting in the point F. I say this shall be the center of the circle. a 10. & 11. 1.

For, let the right lines FA, FB, FC be drawn. Now because AD *b* = DB, and the side DF common, and the angles FDA *c* = FDB, therefore is FB *d* = FA. After the same manner is FC = FA. Therefore a circle described from the center F shall pass through the angles of the triangle given (*viz.*) B, A, C. *Which was to be done.* b *const.* c *const.* & 12. *ax.* d 4. 1.

 Coroll.

Coroll.

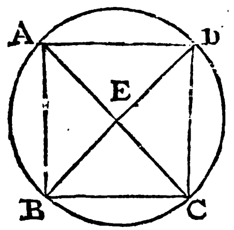
* 31. 3. * Hence, if a triangle be acute-angled, the center ſhall fall within the triangle; if right-angled, in the ſide oppoſite to the right angle, and if obtuſe angled, without the triangle.

Schol.

By the ſame method may a circle be deſcribed, that ſhall paſs through three points given, not being in the ſame ſtrait line.

PROP. VI.

In a circle given EABCD *to inſcribe a ſquare* ABCD.

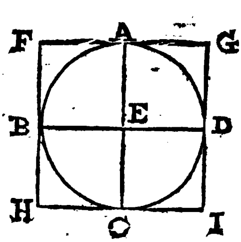


a 11. 1. *a* Draw the diameters AC, BD cutting each other at right angles in the center E. Join the extremes of theſe diameters with the right lines AB, BC, CD, DA. *And the thing is done.*

b 26. 3. c 29. 3. d 31. 3. e 29. def. 1. Now becauſe the four angles at E are right, the *b* arches and *c* ſubtended lines AB, BC, CD, DA, are equal; therefore is the figure ABCD equilateral, and all the angles in ſemicircles, and ſo *d* right. *e* Therefore ABCD is a ſquare inſcribed in a circle given. *Which was to be done.*

PROP. VII.

About a circle given EABCD, *to deſcribe a ſquare* FHIG.



a 17. 3. Draw the Diameters AC, BD, cutting one the other at right angles; through the extremes of theſe diameters *a* draw tangents meeting in F, H, I, G, then I ſay it's done.

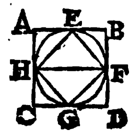
For

For becauſe *b* the angles A and C are right, *c* therefore is FG parallel to HI. After the ſame manner is FH parallel to GI, and therefore FHIG is a Pgr. and alſo right angled. It is equilateral becauſe FG *d* = HI *d* = DB *e* = CA *d* = FH *d* = GI.

Wherefore FHIG is a *f* ſquare circumſcribed to the circle given. *Which was to be done*

b 18. 3.
c 28. 1.
d 34. 1.
e 15. *def.* 1.
f 29. *def.* 1.

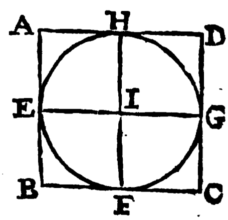
Schol.



A ſquare ABCD deſcribed about a circle is double of the ſquare EFGH inſcribed in the ſame circle.

For the rectangle HB = 2 HEF and HD = 2 HGF by the 41. 1.

PROP. VIII.



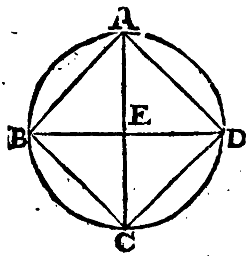
In a ſquare given ABCD, *to inſcribe a circle* IEFGH.

Biſect the ſides of the ſquare in the points H, E, F, G, cutting one the other in I, a circle drawn from the center I thro H ſhall be inſcribed in the ſquare.

For becauſe AH and BF are *a* equal and *b* parallel, *c* therefore is AB parallel to HF, parallel to DC. After the ſame manner is AD parallel to EG, parallel to BC; therefore IA, ID, IB, IC, are parallelograms. Therefore AH *d* = AE *e* = HI = EI = FI = IG. The circle therefore deſcribed from the center I through H, ſhall paſs through H, E, F, G, and touch the ſides of the ſquare ſince the angles H, E, F, G, are right. *Which was to be done.*

a 7. *ax.*
b *hyp.*
c 33. 1.
d 7. *ax.*
e 34. 1.

PROP. IX.

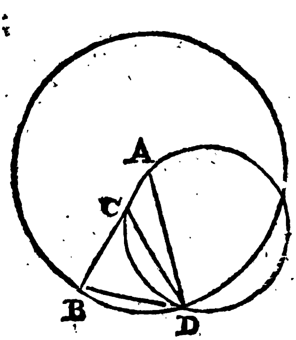


About a square given ABCD, *to describe a circle* EABCD.

Draw the diagonals AC, BD, cutting one the other in E. From the center E through A describe a circle, I say this circle is circumscribed to the square.

a 4. cor. 32. 1. b 6. 1. For the angles ABD and BAC are *a* half of right angles, *b* therefore EA=EB. After the same manner is EA = ED = EC. The circle therefore described from the center E passes through A, B, C, D, the angles of the square given. *Which was to be done.*

PROP. X.



To make an Isosceles triangle ABD, *having each angle at the base* B, *and* ADB *double to the remaining angle* A.

a 11. 2. Take any right line AB, and divide it in C, *a* so that AB x BC may be equal to ACq. From the center A through B, describe the circle ABD; and in this

b 1. 4. circle *b* apply BD = AC, and join AD; I say ABD is the triangle required.

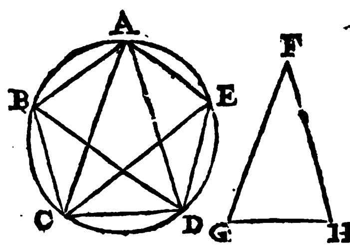
c. 5. 4. For draw DC, and through the points C, D, A, *c*
d 37. 3. draw a circle. Now because AB x BC=ACq = BDq, *d*
e 32. 3. it is evident that BD touches the circle ACD which
f 2. ax. CD cutteth; *e* therefore is the angle BDC=A, and
g 32. 1. therefore the angle BDC + CDA *f* = A + CDA *g* =
h 5. 1. BCD. But BDC + CDA = BDA *h* = CBD, *k*
k 1. ax. therefore the angle BCD = CBD, and therefore DC *l*
l 6. 1. =

$=DB=$ m AC, n wherefore the angle CDA $=$ A m *constr.*
$=$ BDC, therefore ADB $= 2$ A $=$ ABD. *Which was* n 5. 1.
to be done.

Coroll.

Whereas all the angles A, B, D, *h* make up two h 32. 1.
right angles, it's evident that A is one fifth of two right angles.

PROP. XI.



In a circle given ABCDE *to inscribe a Pentagon* ABCDE *equilateral and equiangular.*

a Describe an Isosceles triangle FGH, having each angle at the base double to the other; to the circle, *b* inscribe a triangle CAD equiangular to the said triangle FGH. *c* Bisect the angles at the base ACD and ADC with the right lines DB, CE meeting with the circumference in B and E, join the right lines CB, BA, AE, ED. *Then I say it is done.* a 10. 4. b 2. 4. c 9. 1.

For it is evident by construction, that the angles CAD, CDB, BDA, DCE, ECA, are equal; wherefore the *d* arches and *e* the lines subtending them DC, CB, BA, AE, DE, are equal. Therefore the pentagon is equilateral, and equiangular, *f* because the angles of it BAE, AED, &c. stand on equal *g* arches BCDE, ABCD, &c. d 26. 3. e 29. 3. f 27. 3. g 2. *ax.*

A more easy practice of this problem shall be deliver'd at 10. 13.

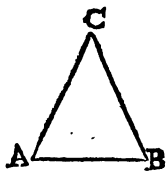
Coroll.

Hence, each angle of an equilateral and equiangular pentagon is equal to three fifths of two right angles, or six fifths of one right angle.

Schol.

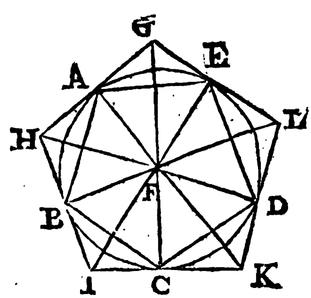
Schol.

Pet. Herig. *Generally all figures whose number of sides is odd, are inscribed in circles by the help of Isosceles triangles, whose angles at the base are multiples of those at the top: and figures whose number of sides is odd, are inscribed in a circle by the help of Isosceles triangles, whose angles at the base are multiples sesquialter of those at the top.*



As in the Isoceless triangle CAB if the angle $A = 3\,C = B$, then will AB be the side of a heptagon. If $A = 4\,C$, then is AB the side of an Enneagone. But if $A = \frac{1}{2}$ C, then is AB the side of a square. And if $A = 2\frac{1}{2}$ C A B will subtend the sixth part of a circumference, and likewise if $A = 3\frac{1}{2}$ then will AB be the side of an octagone.

PROP. XII.



About a circle given FABCDE, *to describe a pentagon* HIKLG, *equilateral and equiangular.*

a 11. 4. *a* Inscribe a pentagon ABCDE in the circle given; and from the center draw the right lines FA, FB, FC, FD, FE; and to those lines draw so many perpendiculars GAH, HBI, ICK, KDL, LEG, meeting in the

b *cor.* 16. 3. points H, I, K, L, G, then I say it is done. For be-

c 2. *cor* 36. cause GA, GE from the same point G *b* touch the circle,

d 8. 1. *c* therefore is $GA = GE$, and *d* therefore the angle GFA $= GFE$.

=GFE, therefore the angle AFE=2GFA. After the same manner is the angle AFH = HFB, and consequently the angle AFB = 2 AFH. *e* But the angle AFE=AFB, *f* therefore the angle GFA = AFH. But also the angle FAH *g*=FAG, and the side FA is common, *h* therefore HA=AG=GE = EL, &c. *k* Therefore HG, GL, LK, KI, IH, the sides of the pentagon are equal, and so also are the angles, because double of the equal angles AGF, AHF, therefore, &c.

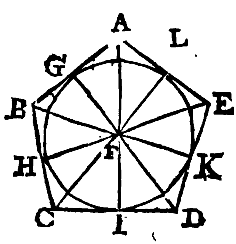
e 27. 3.
f 7 ax.
g 12. ax.
h 26. 1.
k 2. ax.

Coroll.

After the same manner, if any equilateral and equiangled figure be described in a circle, and at the extreme points of the semi-diameters drawn from the center to the angles, be drawn perpendicular lines to the said diameters, I say that these perpendiculars shall make another figure of as many equal sides and equal angles, circumscribed to the circle.

PROP. XIIII.

In an equilateral and equiangular pentagon given ABCDE *to inscribe a circle* FGHK.



a Bisect two angles of the pentagon A and B, with the right lines AI, BK, meeting in the point F. From F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle described from the center F through G will touch all the sides of the pentagon.

a 9. 1.

Draw FC, FD, FE. Because BA *b*=BC, and the side BF common, and the angle FBA *c*=FBC, *d* therefore is AF=FC, and the angle FAB = FCB, but the angle FAB *e*= $\frac{1}{2}$ BAE = $\frac{1}{2}$ BCD. Therefore the angle FCB= $\frac{1}{2}$ BCD. After the same manner are all the whole angles C, D, E bisected. Now whereas the angle FGB *f* = FHB, and the angle FBH = FBG, and the side FB is common, *g* therefore is FG=FH In like manner are all the right lines FH, FI, FK, FL, FG equal. Therefore a circle described from the center F through G passes through the points H, I, K, L, and *h* touches the

b hyp.
c constr.
d 4. 1.
e hyp.
f 12. ax.
g 26. 1.
h cor. 16. 3.

the side of the pentagon, because the angles at those points are right. *Which was to be done.*

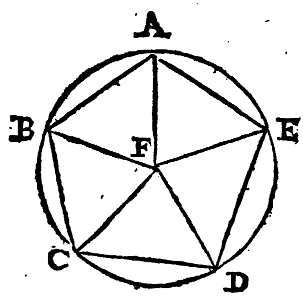
Coroll.

Hence, if any two nearest angles of an equilateral and equiangular figure are bisected, and from that point in which the lines meet that bisect the angles be drawn right lines to the remaining angles of the figure, all the angles of the figure shall be bisected.

Schol.

By the same method may a circle be inscribed in any equilateral and equiangular figure.

PROP. XIV.



About a pentagon given ABCDE *equilateral and equiangular to describe a circle* FABCDE.

Bisect any two angles of the pentagon with the right lines AF, BF, meeting in the point F; the circle described from the center F through A shall be described about the pentagon.

For let FC, FD, FE be drawn. *a* Then the angles C, D, E are bisected; *b* and therefore FA, FB, FC, FD, FE are equal; therefore the circle described from the center F passes through A, B, C, D, E, all the angles of the pentagon. *Which was to be done.*

a cor. 13. 4.
b 6. 1.

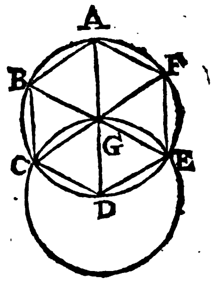
Schol.

Schol.

By the same method is a circle described about any figure which is equilateral and equiangular.

PROP. XV.

In a circle given GABCDEF *to inscribe an Hexagone (or six sided figure)* ABCDEF *equilateral and equiangular.*



Draw the diameter AD; from the center D through the center G describe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. *Then I say it's done*

For the angle CGD *a* = ⅓ of 2 right *a* = DGE *b* = AGF *b* = AGB. *c* Therefore BGC = ⅓ of 2 right = FGE; therefore the *d* arches and *e* subtenses AB, BC, CD, DE, EF, are equal. Therefore the hexagon is equilateral; but it is equiangled also, *f* because all the the angles of it stand upon equal arches.

a *cor.* 32. 1.
b 15. 1.
c *cor.* 13. 1.
d 26. 3.
e 29. 3.
f 27. 3.

Coroll.

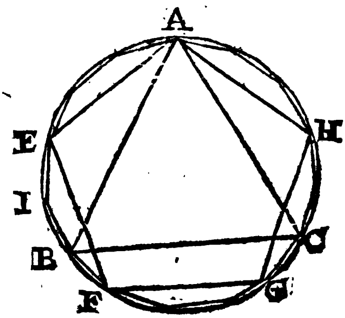
1. Hence, the side of an hexagon inscribed in a circle is equal to the semidiameter.
2. Hereby an equilateral triangle ACE may very easily be described in a circle given.

Schol. Probl.

To make a true hexagon upon a right line given CD. *And. Tacq.*

a Make an equilateral triangle CGD upon the line given CD; from the center G through C and D describe a circle. That circle shall contain the hexagon made upon the given line CD. a 1. 1.

PROP. XVI.



In a circle given AEBC, *to inscribe a quindecagon (or fifteen sided figure) equilateral and equiangular.*

a 11. 4. *a* Inscribe an equilateral pentagon AEFGH in the circle given, and *b* also an equilateral triangle ABC, then

b 2. 4.

I say BF is the side of the quindecagon required.

c *constr.* For the arch AB *c* is $\frac{1}{3}$ or $\frac{5}{15}$ of that periphery whereof AF is $\frac{2}{5}$ or $\frac{6}{15}$, therefore the remaining part BF is $\frac{1}{15}$ of the periphery; and therefore the quindecagon, whose side is BF, is equilateral; but it is equiangular

d 27. 3. also *d* because all the angles insist on equal arches of a circle, whereof every one $\frac{13}{15}$ of the whole circumference. Therefore, &c.

Schol.

A circle is geometrically divided into parts	4, 8, 16, &c. by 6, 4, and 9, 1.
	3, 6, 12, &c. by 15, 4, and 9, 1.
	5, 10, 20, &c. by 11, 4, and 9, 1.
	15, 30, 60, &c. by 16, 4, and 9, 1.

Any other way of dividing the circumference into any parts given is as yet unknown; wherefore in the construction of ordinate figures, we are forced to have recourse to mechanick artifices, concerning which you may consult the writers of practical Geometry.

THE

The FIFTH BOOK OF EUCLIDE's ELEMENTS.

Definitions. April. 16. 1717 DB

I. A Part, is a magnitude of a magnitude, a leſs of a greater, when the leſs meaſureth the greater

II. Multiple is a greater magnitude in reſpect of a leſſer, when the leſſer meaſureth the greater.

III Ratio is the mutual habitude or reſpect of two magnitudes of the ſame kind each to other, according to quantity.

In every ratio that quantity which is referr'd to another quantity is called the antecedent of the ratio, and that to which the other is referr'd is called the conſequent of the ratio, as in the ratio of 6 to 4, 6 is the antecedent and 4 the conſequent.

Note, *The quantity of any ratio is known by dividing the antecedent by the conſequent; as the ratio of 12 to 5 is expreſſed by* $\frac{12}{5}$; *or the quantity of the ratio of* A *to* B *is* $\frac{A}{B}$. *Wherefore, often for brevity ſake we denote the quantities of ratio's thus;* $\frac{A}{B} \sqsubset$, or $=$, or $\sqsupset \frac{C}{D}$. *that is, the ratio of* A *to* B *is greater, equal, or leſs than the ratio of* C *to* D. *Which muſt be well obſerved by thoſe who would underſtand this Book.*

Concerning the divers ſpecies of ratio's, you may pleaſe to conſult interpreters.

IV. Proportion is a ſimilitude of ratio's.

That which is here termed proportion, *is more rightly called* proportionality *or* analogy; *for proportion commonly denotes no more than the ratio betwixt two magnitudes*

V. Thoſe

V. Those numbers are said to have a ratio betwixt them which being multiplied may exceed one the other.

E, 12. | A, 4. B, 6. | G, 24.
F, 30 | C, 10 D, 15. | H, 60.

VI Magnitudes are said to be in the same ratio, the first A to the second B, and the third C, to the fourth D, when the equimultiples E and F of the first A, and the third C compared with the equimultiples G, H, of the second B, and the fourth D, according to any multiplication whatsoever, either both together E, F are less than G H both together, or equal taken together, or exceed one the other together, if those be taken E, G, and F, H, which answer one to the other.

The note hereof is : :; *as* A. B. : : C. D. *That is, as* A *is to* B, *so is* C *to* D. *which signifies that* A *to* B, *and* C *to* D, *are in the same ratio. We sometimes thus express it*

$\frac{A}{B} = \frac{C}{D}$, *that is,* A. B : : C. D.

VII. Magnitudes that have the same ratio (A. B : : C. D. are called proportional.

E, 30. | A, 6. B, 4 | G, 28
F, 60. | C, 12. D, 9 | H, 6.3

VIII When of equimultiples, E the multiple of the first magnitude A exceeds G the multiple of the second B, but F the multiple of the third C exceeds not H the multiple of the fourth D, then the first A to the second B has a greater ratio than the third C to the fourth D.

If $\frac{A}{B} \sqsubset \frac{C}{D}$, *it is not necessary from this definition, that* E *should always exceed* G, *when* F *is less than* H; *but it is granted that this may be.*

IX. Proportion consists in three terms at least. *Whereof the second supplies the place of two.*

X. When three magnitudes A, B, C, are proportional, the first A is said to have a duplicate ratio to the third C, of that it hath to the second B: But when four magnitudes A, B, C, D are proportional, the first A is said to have a triplicate ratio to the fourth D, of what it has to the second B; and so always in order one more, as the proportion shall be extended.

Dupli-

Duplicate ratio is thus expressed $\frac{A}{C} = \frac{A}{B}$ *twice, that is, the ratio of* A *to* C *is duplicate of the ratio of* A *to* B.

Triplicate ratio is thus expressed; $\frac{A}{D} = \frac{A}{B}$ *thrice; that is, the ratio of* A *to* D *is triplicate of the ratio of* A *to* B.

$\div\!\!\cdot$ *Denotes continued proportionals; as* A, B, C, D $\div\!\!\cdot$, *also* 2, 6, 18, 54, $\div\!\!\cdot$ *are continual proportionals.*

XI. Homologous, or Magnitudes of like ratio, are antecedents to antecedents, and consequents to consequents; that is, *if* A. B : : C. D ; A *and* C ; as well as B and D are called homologous magnitudes.

XII. Alternate proportion is the comparing of antecedent to antecedent, and consequent to consequent. *As if* A. B : : C. D. *therefore alternately, or by permutation,* A. C : : B. D. *by the* 16. *of* 5.

In this definition, and the five following, names are given to the six ways of arguing which are often used by Mathematicians: the force of which inferences depends on the propositions of this book, which are named in their explications.

XIII. Inverse ratio is when the consequent is taken as the antecedent, and so compared to the antecedent as the consequent; *as* A. B : : C. D ; *therefore inversly* B. A : : D. C. *by cor.* 4. 5.

XIV. Compounded ratio is when the antecedent and consequent taken both as one are compared to the consequent it self. *As* A. B : : C. D; *therefore by composition* A+B. B : : C+D. D. by 18. 5.

XV. Divided ratio is when the excess wherein the antecedent exceedeth the consequent, is compared to the consequent. *As* A. B : : C. D ; *therefore by division,* A—B. B : : C—D. D, *by* 17. 5.

XVI. Converse ratio is when the antecedent is compared to the excess wherein the antecedent exceeds the consequent. *As* A. B : : C D; *therefore by converse ratio* A. A—B : : C. C—D; *by the coroll. of the* 19. *of the* 5.

XVII. Proportion of equality is where there are taken more magnitudes than two in one order, and also as many magnitudes in another order, comparing two to two being in the same ratio; it cometh to pass that as in the first order of magnitudes, the first is to the last, so in the second order of magnitudes, is the first to

the laſt. Or otherwiſe: it is a compariſon of the extremes together, the mean magnitudes being omitted.

Thus let A, B, C, *be three magnitudes and* D, E, F, *three others, and taking them two by two, let them be in the ſame proportion, that is, let* A. B :: D. E, *and* B. C :: E. F; *now if it be inferr'd that* A *the firſt of the firſt order, is to* C *the laſt, as* D *the firſt of the ſecond order, is to* F *the laſt, this form of arguing is ſaid to be* ex æquo, *or from equality*

XVIII. Ordinate proportion is, when antecedent is to conſequent, as antecedent to conſequent, and as the conſequent is to any other, ſo is the conſequent to any other. *As when* A. B :: D. E. *alſo* B. C :: E. F. *and then it ſhall be* A. C :: D. F. *by the* 22. *of the* 5.

XIX. Perturbate proportion is, when three magnitudes being put, and others alſo, which are equal to theſe in multitude, as in the firſt magnitudes the antecedent is to the conſequent, ſo in the ſecond magnitudes is the antecedent to the conſequent: and as in the firſt magnitudes the conſequent is to any other, ſo in the ſecond magnitudes is any other, to the antecedent. *Thus if* A, B, C, *and* E, F, G, *are two ſets of magnitudes, if* A *the firſt of the firſt ſet, is to* B *the ſecond, as* F *the ſecond of the ſecond ſet, is to* G *the laſt; and alſo if* B *the ſecond of the firſt ſet is to* C *the laſt, as* E *the firſt of the ſecond ſet is to* F *the ſecond, ſuch is called pertubate proportion, and by the* 23. 5. A. C :: E G.

XX. Any number of magnitudes being put; the proportion of the firſt to the laſt is compounded out of the proportions of the firſt to the ſecond, the ſecond to the third, and the third to the fourth, &c. to the laſt.

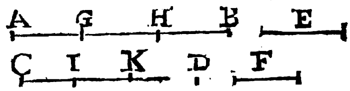
Let there be any number of magnitudes A, B, C, D by this definition $\frac{A}{D} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}$.

Axiom.

Equimultiples of the ſame, or of equal magnitudes are equal to each other.

PROP-

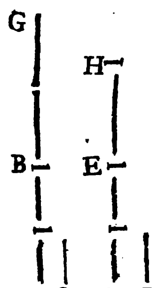
PROP. I.



If there be any number of magnitudes AB, CD, *equimultiples to a like number of magnitudes* E, F, *each to each; whatever multiple one magnitude* AB *is of one* E, *the same multiple is all the magnitudes* AB+CD *to all the other magnitudes* E+F.

Let AG, GH, HB, the parts of the quantity AB, be equal to E, and also let C IK, KD, the parts of the quantity CD be equal to F. The Number of these are put equal to those. Now whereas AG+CI *a* = E+ a 2. ax.
F; *a* and GH+IK=E+F; *a* and HB+KD=E+F, it is evident that AB+CD doth so often contain E+F as one AB contains E. *Which was to be done.*

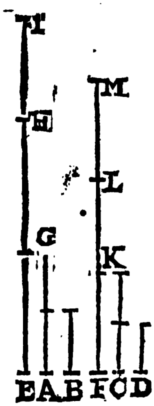
PROP. II.



If the first AB *be the same multiple of the second* C, *as the third* DE *is of the fourth* F, *and if the fifth* BG *be the same multiple of the second* C, *as the sixth* EH *is of the fourth* F; *then shall the first and fifth taken together* (AG) *be the same multiple of the second* C, *as the third and sixth taken together* (DH) *is of the fourth* F.

The number of parts in AB equal each to C is put equal to the numbers of part in DE, whereof each part is equal to F. Likewise the number of parts BG is put equal to the number of parts in EH. Therefore the number of parts in AB+BG is equal to the number of parts in DE+EH. *a* That is, the whole line
AG is the same multiple of C, as the whole line DH is a 2. ax.
of F. *Which was to be demonstrated.*

PROP. III.



If the first A *be the same multiple of the second* B, *as the third* C *is of the fourth* D, *and there be taken* EI, FM *equimultiples of the first and third, then will each of the magnitudes taken be equimultiples, the one* EI *of the second* B, *the other* FM *of the fourth* D.

Let EG, GH, HI, the parts of the multiple EI be equal to A, also let FK, KL, LM, the parts of the multiple FM be equal to C, *a* the number of these is equal to the number of those. Moreover A (that is) EG or GH or HI is put the same multiple of B, as C, or FK, &c. is of D. *b* Therefore EG+GH is the same multiple of the second B, as FK+KL is of the fourth D. *c* By the same way of arguing is EI (EH+HI) the same multiple of B, as FM (FL+LN) is of D. *Which was to be demonstrated.*

a hyp.

b 2. 5.

c 2. 5.

PROP.

PROP. IV.

If the first A *have the same ratio to the second* B, *as the third* C *to the fourth* D; *then also* E *and* F *the equimultiples of the first* A *and the third* C, *shall have the same ratio to* G *and* H *the equimultiples of the second* B *and the fourth* D, *according to any multiplication, if so taken as they answer each to other* (E. G : : F. H.)

Take I and K equimultiples of E and F; and also L and M equimultiples of G and H. *a* Then is I the same multiple of A, as K is of C; *a* and also L is the same multiple of B, as M of D. Therefore whereas it is A. B *b* : : C. D; according to the sixth definition, if I be ⊏, =, ⊐ L, then consequently after the same manner is K ⊏, =, ⊐ M, Therefore when I and K are taken the same multiples of E and F, as L and M of G, and H, then will it be by the seventh definition E. G : : F. H. *Which was to be demonstrated.*

a 3. 5.

b *hyp.*

Coroll.

From hence is wont to be demonstrated the proof of inverse ratio

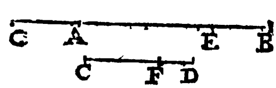
For because A. B : : C. D, therefore if E ⊏, =, ⊐ G, then is *c* likewise F ⊏, =, ⊐ H; therefore it is evident, that if G ⊏, =, ⊐ E, then is H ⊏, =, ⊐ F; *d* therefore B. A : : D. C. *Which was to be demonstrated.*

c 2. *def.* 5.

d 6. *def.* 5.

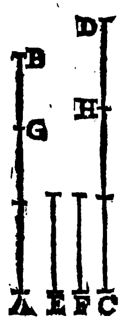
PROP. V.

If a magnitude AB *be the same multiple of a magnitude* CD, *as a part taken from the one* AE *is of a part taken from the other* CF; *the residue of the one* EB, *shall be the same multiple of the residue of the other* FD *as the whole* AB *is of the whole* CD.



Take another GA, which shall be the same multiple of FD the residue, as AB is of the whole CD, or as the part taken away AE, is of the part taken away
a 1. 5. CF. *a* Therefore the whole GA + AE is the same multiple of the whole CF+FD, as the one AE is of
b 6. ax. the one CF, that is, as AB is of CD; therefore GE *b*=
c 3. ax. AB; and *c* so AE which is common being taken away, there remains GA=EB. Therefore, &c.

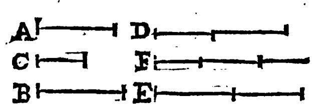
PROP. VI.



If two magnitudes AB, CD, *are equimultiples of two magnitudes* E, F; *and some magnitudes* AG *and* CH *equimultiples of the same* E, F, *be taken away; then the residues* GB, HD, *are either equal to these magnitudes* E, F, *or else equimultiples of them.*

For because the number of parts in AB, whereof each is equal to E, is put equal to the number of parts in CD, whereof each is equal to F; and also the number of parts in AG equal to the number of parts in CH; If from
a 3. ax. one you take AG, and from the other CH, *a* then remains the number of parts in the remainer GB equal to the number of parts in HD; therefore if GB be once E, then is HD once C; if GB be many times E, then is HD so many times C. *Which was to be demonstrated.*

PROP. VII.



Equal magnitudes A *and* B *have the same proportion or ratio to the same magnitude* C. *And one and the same magnitude* C *hath the same ratio to equal magnitudes* A *and* B.

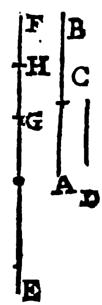
Take D and E equimultiples of the equal magnitudes
a 6. ax. A and B, and F any multiple of C; then is D *a* = E.
b 6 def. 5. Wherefore if D ⊏, =, ⊐ F, then also E will be
c cor. 4. 5. ⊏, =, ⊐ F, *b* therefore A. C: : B. C; and *c* by inversion C. A: : *c* C. B. *Which was to be demonstrated.*

Schol.

Schol.

If instead of the multiple F, two equimultiples be taken, it will be the same way proved that equal magnitudes have the same ratio to any other magnitudes that are equal between themselves.

PROP. VIII.



Of unequal magnitudes AB, AC, *the greater* AB *hath a greater ratio to the same third line* D, *than the lesser* AC; *and the same third line* D *hath a greater ratio to the lesser* AC, *than to the greater* AB

Take EF, EG equimultiples of the said AB, AC, so that EH a multiple of D may be greater than EG, but lesser than EF, (which will easily happen, if both EG and GF be taken greater than D.) It is manifest from 8 *def.* 5. that

$$\frac{AB}{D} \sqsubset \frac{AC}{D} \text{ and } \frac{D}{AB} \sqsupset \frac{D}{AC}.$$

Which was to be demonstrated.

PROP. IX.

Magnitudes which to one and the same magnitude have the same ratio, are equal the one to the other. And if a magnitude have the same ratio to other magnitudes, those magnitudes are equal one to the other.

1. *Hyp.* If A. C : : B. C; I say that A = B. For let A be greater or less than C, *a* then is $\frac{A}{C} \sqsubset$ or $\sqsupset \frac{B}{C}$. *Which is contrary to the Hypothesis.* a 8. 5.

2. *Hyp.* If C. B : : C, A. I say that A = B. For let A be $\sqsubset$ B, *b* then $\frac{C}{B} \sqsubset \frac{C}{A}$. *Which is against the Hypothesis.* b 8 5.

PROP. X.

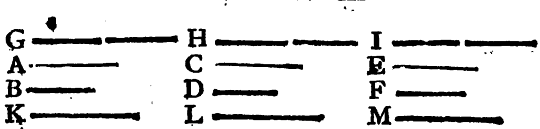
Of magnitudes having ratio to the same magnitude, that which has the greater ratio, is the greater magnitude: and that magnitude to which the same bears a greater ratio, is the lesser magnitude.

1. *Hyp.* If $\frac{A}{C} \sqsubset \frac{B}{C}$. I say that A $\sqsubset$ B.

a 7. 5. For if it be said that A = B, *a* then A.C :: B.C. *Which*
b 8. 5. *is contrary to the Hypothesis.* If A $\sqsupset$ B, *b* then is $\frac{A}{C} \sqsupset \frac{B}{C}$ *Which is also against the Hypothesis.*

2. *Hyp.* If $\frac{C}{B} \sqsubset \frac{C}{A}$. I say that B $\sqsupset$ A; for if you say B = A, it's against the Hypothesis, for it will
c 7. 5. *c* follow that C.B :: C.A. If you say B $\sqsubset$ A, *d* then
d 8. 5. is $\frac{C}{A} \sqsubset \frac{C}{B}$. *Which is also against the Hypothesis.*

PROP. XI.



Proportions which are one and the same to any third, are also the same one to another.

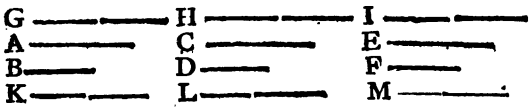
Let A.B :: E. F, and C. D :: E.F. I say that A.B :: C.D. Take G, H, I, equimultiples of A, C, E; and K, L, M, equimultiples of B, D, F. Now *a* be-
a *hyp.* cause A.B :: E. F; if G $\sqsubset$, =, $\sqsupset$ K, *b* then after the same manner I $\sqsubset$, =, $\sqsupset$ M. And likewise *a* because E F :: C.D, if I $\sqsubset$, =, $\sqsupset$ M, *b* then is
b 6 *def.* 5. H likewise $\sqsubset$, =, $\sqsupset$ L. *c* wherefore A.B :: C.D.
c 6. *def.* 5. *Which was to be demonstrated.*

Schol.

Schol.

Proportions that are one and the same to the same proportions, are the same betwixt themselves.

PROP. XII.



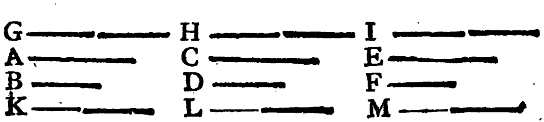
If any number of magnitudes A, B ; C, D; E *and* F *be proportional ; as one of the antecedents* A *is to one of the consequents* B, *so are all the antecedents* A, C, E, *to all the consequents* B, D, F.

Take G, H, I, equimultiples of the antecedents, and K, L, M, of the consequents. Because that one G is the same multiple of one A, *a* as all G, H, I, are of all A, C, E ; and because one K is the same multiple of one B, as all K, L, M, are of all B, D, F: Moreover because A. B *b* : : C. D *b* : : E. F, if G be $\sqsubset$, $=$, or $\sqsupset$ K, then will H likewise be $\sqsubset$, $=$, $\sqsupset$ L, and I $\sqsubset$, $=$, $\sqsupset$ M ; and so if G $\sqsubset$, $=$, $\sqsupset$ K, in like manner will G + H + I be $\sqsubset$, $=$, $\sqsupset$ K + L + M ; *c* wherefore A. B : : A + C + E. B + D + F. *Which was to be demonstrated.* a 1. 5. b *hyp.* c 6. *def.* 5.

Coroll.

Hence, if like proportionals be added to like proportionals, the wholes shall be proportional.

PROP. XIII.



If the first A *have the same ratio to the second* B, *that the third* C *hath to the fourth* D, *and if the third* C *have a greater proportion to the fourth* D, *than the fifth* E *to the*

the sixth F; *then also shall the first* A *have a greater proportion to the second* B, *than the fifth* E *to the sixth* F.

Take G, H, I, equimultiples of A, C, E, and K, L, M, equimultiples of B, D, F. Now because that A.
a 6. *def.* 5. B : : C D, if H $\sqsubset$ L, *a* then is G $\sqsubset$ K; but because
b 8. *def.* 5. $\frac{C}{D} \sqsubset \frac{E}{F}$, *b* it may be that H $\sqsubset$ L, and yet I not $\sqsubset$
c 8. *def.* 5. M. *c* Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$. *Which was to be demon.*

Schol.

But if $\frac{C}{D} \sqsupset \frac{E}{F}$, then also is $\frac{A}{B} \sqsupset \frac{E}{F}$. Also, if $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if $\frac{A}{B} \sqsupset \frac{C}{D} \sqsupset \frac{E}{F}$, then is $\frac{A}{B} \sqsupset \frac{E}{F}$.

PROP. XIV.

A B C D

If the first A *have the same ratio to the second* B, *that the third* C *hath to the fourth* D; *and if the first* A *be greater than the third* C; *then shall the second* B *be greater than the fourth* D. *But if the first* A *be equal to the third* C, *then the second* B *shall be equal to the fourth* D; *but if* A *be less, then is* B *also less.*

a 8. 5. Let A $\sqsubset$ C; *a* then $\frac{A}{B} \sqsubset \frac{C}{B}$ *b* but
b *hyp.*
c. 13. 5. $\frac{A}{B} = \frac{C}{D}$; *c* therefore $\frac{C}{D} \sqsubset \frac{C}{B}$. *c* therefore B $\sqsubset$ D. By the like way of argument, if A $\sqsupset$ C,
d 10 5. *d* then is B $\sqsupset$ D But if A be put equal to C, then
e 7. 5. C. B : : *e* A. B *f* : : C. D. *g* Therefore B = D. *Which*
f *hyp.* *was to be demonstrated.*

g 11. 5. &
9 5.

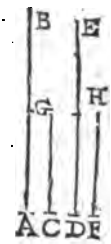
Schol.

Schol.

By an argument à *fortiori*, if $\frac{A}{B} \sqsupset \frac{C}{D}$, and A $\sqsubset$ C, then is B $\sqsubset$ D. Likewise if A = B, then is C = D, and if A $\sqsubset$, or $\sqsupset$ B, then also is C $\sqsubset$ or $\sqsupset$ D.

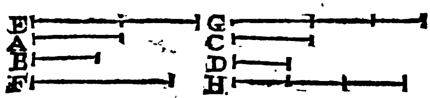
PROP. XV.

Parts C *and* F *are in the same ratio, with their like multiples* AB *and* DE, *if taken correspondently.* (AB. DE : : C. F.)



Let AG, GB, parts of the multiple AB be equal to C; and let DH, HE, parts of the multiple DE be equal to F. *a* The number of these parts is equal to the number of those. Therefore whereas *b* AG. C : : DH. F, and GB. C : : HE. F; therefore is *c* AG + GB (AB) DH + HE (DE) : : C. F. *Which was to be demonstrated.* a *hyp.* b 7. 5. c 12. 5.

PROP. XVI.



If four magnitudes A, B, C, D, *are proportional, they also shall be alternately proportional* (A. C : : B. D)

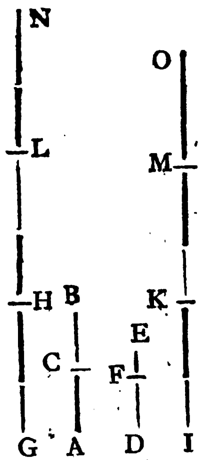
Take E and F equimultiples of A and B; take also G and H equimultiples of C and D. Therefore E. F *a* : : A. B *b* : : C. D *a* : : G. H. Wherefore if E $\sqsubset$, =, $\sqsupset$ *c* G, then likewise is F $\sqsubset$, =, $\sqsupset$ H *d* Therefore A. C : : B. D. *Which was to be demonstrated.* a 15. 5. b *hyp.* c 11. 5. & 14 5. d 6 *def.* 5.

Schol.

Alternate ratio has place only when the quantities are of the same kind. For heterogeneous quantities are not compared together.

PROP.

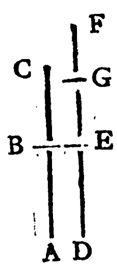
PROP. XVII.



If magnitudes compounded are proportional (AB. CB : : DF. FE.) *they shall be proportional also when divided.* (AC. CB : : DF. FE.)

Take GH, HL, IK, KM, in order equimultiples of AC, CB, DF, FE; and also LN, MO, equimultiples of CB, FE. The
a 1. 5. whole GL is *a* the same multiple of the whole AB, as one
b *constr.* GH is of one AC, *b* that is, as
c 1. 5. IK of DF, *c* or as the whole IM of the whole DE. Also HN
d 2. 5. (HL+LN) is *d* the same multiple of CB, as KO, (KM+MO) is of FE. Therefore, whereas by Hyp. AB. BC : : DE. EF, if GL be $\sqsubset$, $=$, $\sqsupset$ HN, then likewise *e* will IM $\sqsubset$, $=$, $\sqsupset$ KO. Take
e 6. *def.* 5. from these HL, KM, that are equal; and if the remainder GH be $\sqsubset$, $=$, $\sqsupset$ LN, *f* then will IK
f 5. *ax.* $\sqsubset$, $=$, $\sqsupset$ MO, *g* whence AC. CB : : DF. FE. *Which*
g 6. *def.* 5. *was to be demonstrated.*

PROP. XVIII.

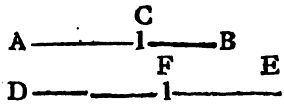


If magnitudes divided are proportional (AB. BC : : DE. EF,) *the same also being compounded shall be proportional* (AC. CB : : DF. FE.)

For if it can be, let AC. CB : : DF. FG $\sqsupset$ FE. *a* Then by division will
a 17. 5. AB. BC : : DG. GF; *b* that is, DG. GF
b *hyp.* & 11. 5. : : DE. EF, and since DG $\sqsubset$ DE,
c 14. 5. *c* therefore is GF $\sqsubset$ EF. *Which is*
d 9. *ax.* *absurd.* The like absurdity will follow if it be said AC. CB : : DF. GF $\sqsubset$ FE.

PROP.

PROP. XIX.



If the whole AB *be to the whole* DE *as the part taken away* AC *is to the part taken away* DF, *then shall the residue* CB *be to the residue* FE, *as the whole* AB *is to the whole* DE.

Because *a* AB. DE : : AC. DF, *b* therefore by permutation AB. AC : : DE. DF, *c* and thence by division AC. CB : : DF. FE; *b* wherefore again by permutation AC. DF :: CB. FE; *d* that is, AB. DE : : CB. FE. *Which was to be demonstrated.*

a *hyp.*
b 16. 5.
c 17. 5.
d *hyp.* & 11. 5.

Coroll.

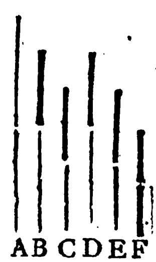
Hence, If like proportionals be subtracted from like proportionals, the residues shall be proportional.

2. *Hence is converse ratio demonstrated.*

Let AB. CB : : DE. FE. I say that AB. AC : : DE. DF. For by *a* permutation AB. DE : : CB. FE, *b* therefore AB. DE : : AC. DF, whence again by permutation AB. AC : : DE. DF. *Which was to be demonstrated.*

a 16. 5.
b 19. 5.

PROP. XX.



If there are three magnitudes A, B, C, *and others* D, E, F, *equal to those in number, which being taken two and two in each order are in the same ratio,* (A. B : : D. E; and B. C : : E. F,) *and if* ex æquo *the first* A *be greater than the third* C; *then shall the fourth* D *be greater than the sixth* F. *But if the first* A *be equal to the third* C, *then the fourth* D *shall be equal to the sixth* F; *and if* A *be less than* C, *then shall* D *be less than* F.

1. *Hyp.* If $A \sqsubset C$. Because *a* E. F : : B. C, by *b* inversion it shall be F. E : : C. B. *c* But $\frac{C}{B} \sqsupset \frac{A}{B}$, *d* therefore

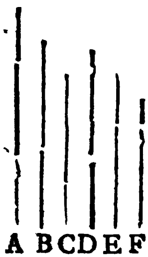
a *hyp.*
b *cor.* 4. 5.
c *hyp.* & 8. 5.
d *sch.* 13. 5.

e 10. 5. fore $\frac{F}{E} \sqsupset \frac{A}{B}$ or $\frac{D}{E}$, *e* therefore D $\sqsubset$ F. *Which was to be demonstrated.*

2. *Hyp.* By the same way of arguing, if A $\sqsupset$ C, it will appear that D $\sqsupset$ F.

f 7. 5. 3. *Hyp.* If A $=$ C. Because F.E : : C.B : : *f* AB : :
g 11. 5 & 9. 5. D.E, *g* therefore is D $=$ F. *Which was to be dem.*

P. ROP. XXI.



If there are three magnitudes A, B, C, *and others also* D, E, F, *equal to them in number, which taken two and two are in the same ratio; and their proportion perturbate* (A.B : : E.F, *and* B.C. : : D.E.) *and if* ex æquo *the first* A *be greater than the third* C, *then is the fourth* D *greater than the sixth* F; *but if the first be equal to the third, then is the fourth equal to the sixth; if less, so is the other likewise.*

a hyp. 1 *Hyp.* If A $\sqsubset$ C; then because *a* D. E : : B. C,
b 8. 5. therefore inversely E. D : : C. B, but $\frac{C}{B}$ *b* $\sqsupset$ $\frac{A}{B}$;
c sch. 13. 5. d 10. 5. *c* therefore $\frac{E}{D} \sqsupset \frac{A}{B}$, that is, than $\frac{E}{F}$, *d* therefore D $\sqsubset$ F.

2. *Hyp.* By the like argument, if A $\sqsupset$ C, then is D $\sqsupset$ F.

e 7. 5. f hyp. g 9. 5. 3. *Hyp.* If A $=$ C; then because E. D : : *e* C. B : : *e* A.B : : *f* E. F, *g* therefore is D $=$ F. *Which was to be demonstrated.*

PROP.

PROP. XXII.

If there be any number of magnitudes A, B, C, *and others equal to them in number* D, E, F, *which taken two and two are in the same ratio* (A. B :: D. E *and* B C :: E. F.) *they shall be in the same ratio also by equality* (A.C :: D.F.)

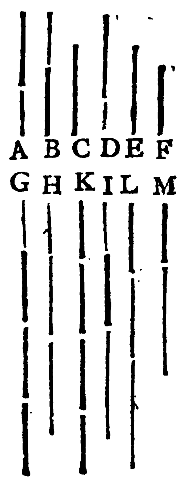
Take G, H, equimultiples of A, D; and I, K, of B, E; and also L, M, of E. F.

Because *a* A. B :: D. E, *b* therefore G. I :: H. K; and in like manner I. L :: K. M. therefore, if G $\sqsubset$, =, $\sqsupset$ L, *c* then is H $\sqsubset$, =, $\sqsupset$ M, *d* therefore A. C :: D. F. By the same way of demonstration if further C. N :: F. O, then by equality A. N :: D.O. *Which was to be demonstrated.*

a *hyp.*
b 4. 5.
c 20. 5.
d 6. *def.* 5.

PROP. XXIII.

If there are three magnitudes A, B, C, *and others* D, E, F, *equal to them in number, which taken two and two are in same ratio, and their proportion perturbate* (A.B :: E. F, *and* B.C :: D. E) *they shall be in the same ratio also by equality* (A. C :: D. F.)



Take G, H, I, equimultiples of A, B, D; and also K, L, M, equimultiples of C, E, F. Then G. H :: *a* A.B :: *b* E.F. *a* :: L. M. Moreover because *b* B. C :: D E, thence is *c* H. K :: I. L; therefore G, H, K, and I, L, M, are as in 21. 5. Therefore if G be $\sqsubset$, =, $\sqsupset$ K, then is likewise I $\sqsubset$, =, $\sqsupset$ M, and so *d* consequently A.C :: D. F. *Which was to be demonstrated.*

a 15. 5.
b *hyp.*
c 4. 5.
d 6. *def.* 5.

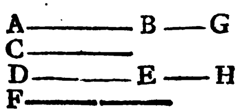
If

If there are more magnitudes than three, this way of demonstration holds good in them also.

Coroll.

* 22 & 23. 5. and 20. def. 1. From hence * it follows, that ratio's compounded of the same ratio's, are among themselves the same; as also that the same parts of the same ratio's, are among themselves the same.

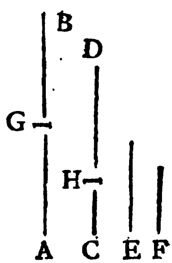
PROP. XXIV.



If the first magnitude AB, *has the same ratio to the second* C, *which the third* DE, *has to the fourth* F; *and if the fifth* BG *has the same ratio to the second* C, *which the sixth* EH *has to the fourth* F; *then shall the first compounded with the fifth* (AG) *have the same ratio to the second* C, *which the third compounded with the sixth* (DH) *has to the fourth* F.

a hyp. b 22 5. c hyp. For because *a* AB. C : : DE. F, and by the Hyp. and inversion C. BG : : F. EH; therefore by *b* equality AB. BG : : DE. EH, whence by compounding, AG. BG : : DH. EH. Also *c* BG. C : : EH. F. Therefore again by *b* equality AG. C : : DH. F. *Which was be demonstrated.*

PROP. XXV.



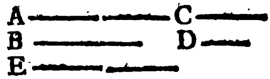
If four magnitudes are proportional (AB. CD : : E. F) *the greatest* AB *and the least* F *shall be greater than the rest* C, D, *and* E.

a hyp. b 7. 5. c 19. 5. d hyp. e sch. 14 5 Make AG=E, and CH=F. Because AB. CD : : *a* EF. *b* AG. CH, *c* thence is AB. CD : : GB. HD; *d* but AB ⊏ CD, *e* therefore GB ⊏ HD. But AG + F=E + CH, therefore AG + F + GB ⊏ E + CH + HD that is, AB + F ⊏ E + CD. *Which was to be demonstrated.*

These propositions which follow are not Euclide's, *but taken out of other Authors, and here subjoyned because of their frequent use.*

PROP.

PROP. XXVI.

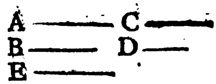


If the first have a greater proportion to the second, than the third to the fourth; then by conversion, the second shall have a less proportion to the first, than the fourth to the third.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, I say that $\frac{B}{A} \sqsupset \frac{D}{C}$. For conceive $\frac{C}{D} = \frac{E}{B}$; *a* therefore $\frac{A}{B} \sqsubset \frac{E}{B}$; *b* whence $A \sqsubset E$, *c* therefore $\frac{B}{A} \sqsupset \frac{B}{E}$, *d* or $\frac{D}{C}$: *Which was to be demonstrated.*

a 13. 5.
b 10. 5.
c 8. 5
d cor. 4. 5.

PROP. XXVII.

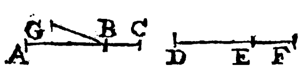


If the first have a greater proportion to the second, than the third to the fourth; then alternately the first shall have a greater proportion to the third, than the second to the fourth.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, then I say $\frac{A}{C} \sqsubset \frac{B}{D}$. For conceive $\frac{E}{B} = \frac{C}{D}$, *a* therefore $A \sqsubset E$, *b* and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$. *c* or $\frac{B}{D}$. *Which was to be demonstrated.*

a 10. 5.
b 1. 5.
c 16 5.

PROP. XXVIII.

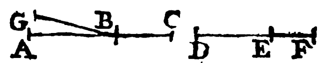


If the first have a greater proportion to the second than the third to the fourth, then the first compounded with the second shall have a greater proportion to the second, than the third compounded with the fourth, to the fourth.

G Let

Let $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. I say that $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$: For
a 10. 5. conceive $\frac{GB}{BC} = \frac{DE}{EF}$, *a* therefore is, $AB \sqsubset GB$, add
b 4. ax. BC to each part, then *b* will $AC \sqsubset GC$, *c* therefore
c 8. 5.
d 18. 5. $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$, *d* that is, $\frac{DF}{EF}$. *Which was to be dem.*

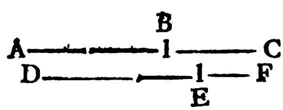
PROP. XXIX.



If the first compounded with the second has a greater proportion to the second, than the third compounded with the fourth hath to the fourth; then by division the first shall have a greater proportion to the second, than the third to the fourth.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$, then I say $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. For
a 16. 5. conceive $\frac{GC}{BC} = \frac{DF}{EF}$, *a* therefore $AC \sqsubset GC$. Take
b 5. ax. away BC, which is common; then *b* remains $AB \sqsubset$
c 8. 4.
d 17. 5. GB; *c* therefore $\frac{AB}{BC} \sqsubset \frac{GB}{BC}$ *d* or $\frac{DE}{EF}$. *Which was to be demonstrated*

PROP. XXX.



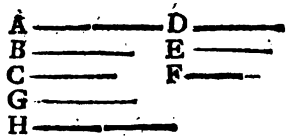
If the first compounded with the second, has, a greater proportion to the second, than the third compounded with the fourth, hath to the fourth, then by converse ratio shall the first compounded with the second have a lesser ratio to the first, than the third compounded with the fourth shall have to the third.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. Then I say that $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$.

For

For becauſe that $\frac{AC}{BC}$ *a* $\sqsubset$ $\frac{DF}{EF}$, *b* therefore by diviſion a *hyp.*
b 29. 5.
$\frac{AB}{BC} \sqsubset \frac{DE}{EF}$, by converſion *c* therefore $\frac{BC}{AB} \sqsupset \frac{EF}{DE}$ c 26. 5.
and *d* therefore by compoſition $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. *Which* d 28. 5.
was to be demonſtrated.

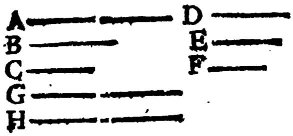
PROP. XXXI.



If there are three magnitudes A, B, C, *and others alſo* D, E, F, *equal to them in number; and if there be a greater proportion of the firſt of the former to the ſecond, than there is of the firſt of the laſt to their ſecond* $\left(\frac{A}{B} \sqsubset \frac{D}{E};\right)$ *and there be alſo a greater proportion of the ſecond of the firſt magnitudes to the third, than there is of the ſecond of the laſt magnitudes to their third* $\left(\frac{B}{C} \sqsubset \frac{E}{F}.\right)$ *Then by equality alſo ſhall the ratio of the firſt of the former magnitudes to the third, be greater than the ratio of the firſt of the latter magnitudes, to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}.\right)$

Conceive $\frac{G}{C} = \frac{E}{F}$, *a* therefore is B $\sqsubset$ G, and *b* there- a 10. 5.
b 8. 5.
fore $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{E}$, *c* therefore c 13. 5.
$\frac{H}{G} \sqsupset \frac{A}{B}$; therefore much more $\frac{H}{G} \sqsupset \frac{A}{G}$, *d* wherefore d 10. 5.
A $\sqsubset$ H, *e* and conſequently $\frac{A}{C} \sqsubset \frac{H}{C}$, *f* or $\frac{D}{F}$. e 8. 5.
f 22. 5.

PROP. XXXII.



If there be three magnitudes A, B, C, *and others* D, E, F, *equal to them in number; and there be a greater proportion of the first of the former magnitudes to the second, than there is of the second of the latter to the third* $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ *and also the ratio of the second of the former to the third be greater than the ratio of the first of the latter to the second* $\left(\frac{B}{C} \sqsubset \frac{D}{E}\right)$ *then by equality also shall the proportion of the first of the former to the third, be greater than that of the first of the latter to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}.\right)$

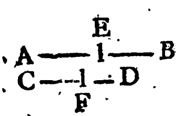
a 10. 5. Suppose $\frac{G}{C} = \frac{D}{E}$ therefore is *a* B $\sqsubset$ G, and there-

b 8. 5. c *schol.* 13. 5 fore *b* $\frac{A}{G} \sqsubset \frac{A}{B}$. Again, Suppose $\frac{H}{G} = \frac{E}{F}$; therefore is *c*

$\frac{H}{G} \sqsupset \frac{A}{G}$, and consequently *a* A $\sqsubset$ H, and thence *b*

d 13. 5. $\frac{A}{C} \sqsubset \frac{H}{C}$ *d* or $\frac{D}{F}$. *Which was to be demonstrated.*

PROP. XXXIII.



If the proportion of the whole AB *to the whole* CD *be greater than the proportion of the part taken away* AE *to the part taken away* CF; *then shall also the ratio of the remainder* EB *to the remainder* FD *be greater than that of the whole* AB *to the whole* CD.

Because

Because that $\frac{AB}{CD}$ a $\sqsubset$ $\frac{AE}{CF}$, b therefore by permutation $\frac{AB}{AE} \sqsubset \frac{CD}{CF}$; c therefore by converse ratio $\frac{AB}{EB} \sqsupset \frac{CD}{FD}$, and by permutation again $\frac{AB}{CD} \sqsupset \frac{EB}{FD}$; a hyp. b 27. 5. c 32. 5.

Which was to be demonstrated.

PROP. XXXIV.

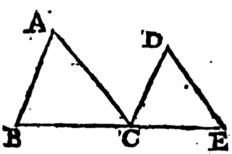
If there be any number of magnitudes, and others also equal to them in number; and the proportion of the first of the former to the first of the latter be greater than that of the second to the second, and that greater than the proportion of the third to the third, and so forward: all the former magnitudes together shall have a greater ratio to all the latter together, than all the former, leaving out the first, shall have to the latter, leaving out the first; but less than that of the first of the former to the first of the latter; and lastly, greater than that of the last of the former to the last of the latter.

You may please to consult Interpreters for the demonstration hereof, we having for brevity sake omitted it, and because 'tis of no use in these Elements.

The End of the fifth Book.

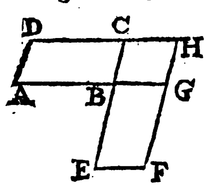
The Sixth Book OF EUCLIDE's ELEMENTS.

Definitions.



I. LIke right-lined figures (ABC, DCE) are ſuch whoſe ſeveral angles are equal one to the other, and alſo their ſides about the equal angles, proportional.

The angle B = DCE, *and* AB. BC :: DC. CE. *Alſo the angle* A = D, *and* BA. AC :: CD. DE. *Laſtly the angle* ACB = E, *and* BC. CA :: CE. ED.

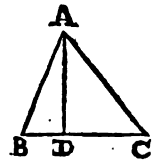


II Reciprocal figures are (BD, BF) when in each of the figures there are terms both antecedent and conſequent (that is, AB. BG :: EB. BC.)

III. A right line AB is ſaid to be cut according to extreme and mean proportion, when as the whole AB is to the greater ſegment AC, ſo is the greater ſegment AC to the leſs CB (AB. AC :: AC. CB)

IV. The

IV. The altitude of any figure ABC, is a perpendicular line AD drawn from the top A to the base BC.

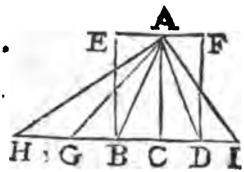


V. A ratio is said to be compounded of ratio's, when the quantities of the ratio's, being multiplied into one another, do produce a ratio. *As the ratio of* A *to* C *is compounded of the ratio's of* A *to* B *and* B *to* C. *For* $\frac{A}{B} \times \frac{B}{C}$

$$a = \frac{AB}{BC} \quad b = \frac{A}{C}.$$ a 20. *def*. 5. b 15. 5.

PROP I.

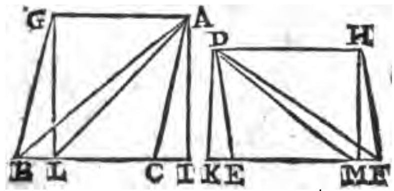
Triangles ABC, ACD, *and parallelograms* BCAE, CDFA, *which have the same height, are to each other, as their bases*, BC, CD.



a Take as many as you please, BG, GH, equal to to BC, and also DI = CD, and join AG, AH, AI. a 3. 1.

b The triangles ACB, ABG, AGH, are equal, and *b* also the triangle ACD = ADI. Therefore the triangle ACH is the same multiple of the triangle ACB, as the base HC is of the base BC; and the triangle ACI the same multiple of the triangle ACD, as the base CI is of CD. But if HC ⊏, =, ⊐ CI, *c* then is likewise the triangle AHC ⊏, =, ⊐ ACI; and *d* therefore BC. CD : : the triangle ABC. ACD : : *e* Pgr. CE. CF, *Which was to be demonstraed.* b 38. 1. c *sch*. 38. 1. d 6. *def*. 5. e 41. 1. & 15. 5

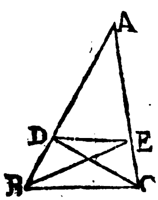
Schol.



Hence *triangles*, ABC, DEF, *and Pgrs.* AGBC, DEFH, *whose bases* BC, EF *are equal, are to each other as their altitudes*, AI, DK.

a 3. 1. *a* Take IL = CB, and KM = EF; and join LA,
b 7. 5. LG, MD, MH. then is it evident, that the triangle
c 1. 6. ABC DEF : : *b* ALI. DKM : : *c* AI. DK : : *d* Pgr.
d 41. 1 & AGBC. DEFH. *Which was to be demonstrated.*
15 5.

PROP. II.

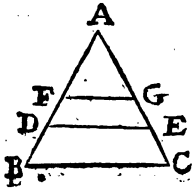


If to one side BC *of a triangle* ABC, *be drawn a parallel right line* DE, *the same shall cut the sides of the triangle proportionally* (AB. BD : : AE EC.) *And if the sides of the triangle are cut proportionally* (AD. BD : AE. EC) *then a right line* DE, *joining the points of section* D, E, *shall be parallel to* BC, *the other side of the triangle.* Draw CD and BE.

1. *Hyp.* Because the triangle DEB *a* = DEC, *b* there-
a 37. 1. fore shall be the triangle ADE. DBE : : ADE. ECD.
b 7. 5. But the triangle AED. DBE : : *c* AD, DB, and the tri-
c 1. 6. angle ADE. DEC : : AE. EC, *d* therefore AD. DB : :
d 11. 5. AE. EC.

e 1. 6. 2. *Hyp.* Because AD. DB : : AE. EC, *e* that is the
f 9. 5. triangle ADE DBE : : ADE. ECD; *f* therefore is the
g 39. 1. triangle DBE = ECD; and *g* therefore DE, BC are parallels. *Which was to be demonstrated.*

Schol.



If there are drawn several lines DE, FG parallel to one side BC of a triangle, all the segments of the sides shall be proportional.

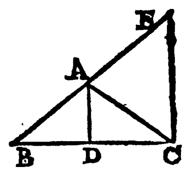
For

For DF. FA *a* : : EG. GA ; and compounding and inverting, FA. DA : : GA. EA ; *a* and DA. DB : : EA. EC, therefore by equality DF. DB : : EG. EC. *Which was to be demonstrated.* a 2 6.

Coroll.

If DF. DB : : EG. EC ; *a* then BC, DE, FG, shall be parallels.

PROP. III.



If an angle BAC *of a triangle* BAC *be bisected, and the right line* AD, *that bisects the angle, cut the base also ; then shall the segments of the base have the same ratio that the other sides of the triangle have,* (BD. DC : : AB. AC) *And if the segments of the base have the same ratio, that the other sides of the triangle have* (BD. DC : : AB. AC) *then a right line* AD *drawn from the top* A *to the section* D, *shall bisect that angle* BAC *of the triangle.*

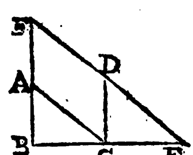
Produce BA, and make AE = AC, and join CE.

1. *Hyp.* Because AE = AC, therefore is the angle ACE *a* = E *b* = half BAC *c* = DAC ; *d* therefore DA, CE are parallels. *e* Wherefore BA. AE (AC) : : BD DC.

2. *Hyp.* Because BA. AC (AE) : : BD. DC, *f* therefore are DA, CE parallels ; and *g* therefore is the angle BAD = E ; and the angle DAC *g* = ACE *h* = E, *k* therefore the angle BAD = DAC. Wherefore the angle BAC is bisected. *Which was to be demonstrated.*

a 5. 1.
b 32. 1.
c *hyp.*
d 27. 1.
e 2. 6.
f 2. 6.
g 29. 1.
h 5. 1.
k 1. *ax.*

PROP. IV.



Of equiangular triangles ABC, DCE, *the sides are proportional which are about the equal angles,* B, DCE, (AB. BC : : DC. CE, &c) *And the sides* AB, DC, &c *which are subtended under the equal angles* ACB, E, &c. *are homologous, or of like ratio.*

Set the side BC in a direct line to the side CE, and produce BA and ED till they *a* meet. a 32. 1. & 13. *ax.*

Be-

b *hyp.* Because the angle B *b* = ECD, *c* therefore BF,
c 28. 1. CD are parallel: Also because the angle BCA *b* = CED, *c* therefore are CA, EF parallel. Therefore the figure CAFD is a Pgr. *d* therefore AF = CD, and AC
d 34. 1. = *d* FD. Whence it is evident, that AB. AF (CD) : : *e*
e 2. 6. BC. CE. *f* by permutation therefore AB. BC :: CD. CE,
f 16. 5. also BC. CE : : FD (AC.) DE. *f* and thence by permu-
g 22. 5. tation BC. AC : : CE. DE. *g* Wherefore also by equality AB. AC : : CD. DE. Therefore, &c.

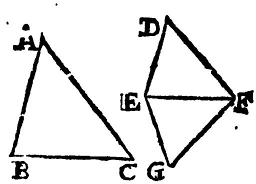
Coroll.

Hence AB. DC : : BC. CE : : AC. DE.

Schol.

Hence, if in a triangle FBE there be drawn AC a parallel to one side FE, the triangle ABC shall be like to the whole FBE.

PROP. V.



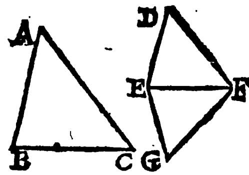
If two triangles ABC, DEF, *have their sides proportional* (AB. BC : : DE. EF, *and* AC. BC : : DF. EF, *and also* AB. AC : : DE. DF) *those triangles are equiangular, and those angles equal under which are subtended the homologous sides.*

a 23. 1. At the side EF *a* make the angle FEG = B, and the
b 32. 1. angle EFG = C; *b* whence the angle G = A. There-
c 4. 6. fore GE. EF *c* : : AB. BC : : *d* DE. EF. *e* and therefore
d *hyp.* GE = DE. Likewise GF. FE *c* : : AC. CB : : *d* DF.
e 11. 5. & FE, *e* therefore GF = DF. Therefore the triangles
9. 5. DEF, GEF, are mutually equilateral. *f* Therefore
f 8. 1. the angle D = G = A, and the angle FED *f* = FEG
g 32. 1. = B, and *g* consequently the angle DFE = C. Therefore, &c.

PROP.

PROP. VI.

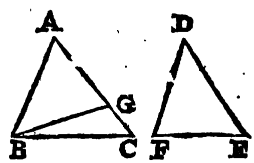
If two triangles ABC, DEF *have one angle* B *equal to one angle* DEF, *and the sides about the equal angles* B, DEF *proportional* (AB. BC : : DE. EF) *then those triangles* ABC, DEF, *are equiangular, and have those angles equal, under which are subtended the homologous sides.*



At the side EF make the angle FEG = B, and the
angle EFG = C; *a* then will the angle G = A. There- a 32. 1.
fore GE. EF : : *b* AB. BC : : *c* DE. EF, *d* and therefore b 4. 6.
DE = GE. But the angle DEF *e* = B *f* = GEF; c *hyp.*
therefore the angle D *g* = G = *A*, *h* and consequently d 9. 5.
the angle EFD = C. *Which was to be demonstrated.* e *hyp.*
f *constr.*
g 4. 1.
h 32. 1.

PROP. VII.

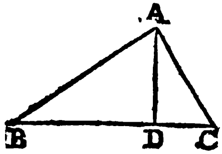
If two triangles ABC, DEF *have one angle* A *equal to one angle* D, *and the sides about the other angles* ABC, E, *proportional* (AB. BC : : DE. EF) *and if they have the remaining anlges* C, F, *either both less or both not less than a right angle; then shall the triangles* ABC, DEF, *be equiangular, and have those angles equal about which the proportional sides are.*



For, if it can be, let the angle ABC $\sqsubset$ E, and
make the angle ABG = E. Therefore, whereas the
angle A *a* = D *b* thence is the angle AGB = F. a *hyp.*
Therefore AB. BG *c* : : DE. EF : : *d* AB. BC, *e* therefore b 32. 1.
BG = BC, *f* therefore the angle BGC = BCG. *g* There- c 4. 6.
fore BGC or C is less than a right angle, and *h* conse- d *hyp.*
quently AGB or F is greater than a right: Therefore e 9. 5.
the angles C and F are not of the same species or kind, f 5. 1.
which is against the Hypothesis. g *cor.* 17 1.
h *cor.* 13. 1.

PROP.

PROP. VIII.



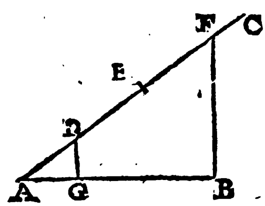
If a line AD be drawn from the right angle A, of a right angled triangle ABC, perpendicular to the base BC; then the triangles ADB, ADC, on each side the perpendicular, are like both to the whole ABC, and to one another,

a hyp. b 12. ax. c 32, 1. & 4. 6. d vid. 21. 6. For because BAC, ADB are *a* right angles, *b* and so equal, and B common; the triangles BAC, ADB, *c* are like. By the same way of arguing BAC, ADC, are like; *d* whence also ADB, ADC will be like. *Which was to be demonstrated.*

Coroll.

e 1. def. 6. Hence, 1. $BD.DA$ *e* :: $DA.DC$.
2. $BC.AC$ :: $AC.DC$, and $CB.BA$ :: $BA.BD$.

PROP. IX

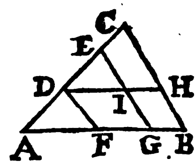


From a right line given AB to cut off any part required, as one third (AG)

a 3. 1. From the point A draw an infinite line AC any wise, in which *a* take any three equal parts AD, DE, FE, join FB, to which from

b 31. 1. c 2. 6. d 18. 5. D *b* draw the parallel DG; and the thing is done.
For $GB.AG$ :: *c* $FD.AD$; whence by *d* composition $AB.AG$ :: $AF.AD$, therefore since $AD =$ one third of AF, shall AG be $=$ one third of AB. *Which was to be done.*

PROP. X.

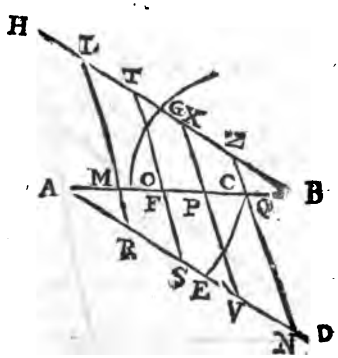


To divide a given undivided right line AB (*in* F, G,) *as another given right line is divided* (*in* D, E.)

Let a right line BC join the extremities of the line divided, and of the line not divided; and parallel to this, from the points E, D, *a* draw EG, DF, meeting with the right line which is to be cut in G and F; then the thing is done. a 31. 1.

For let DH be *a* drawn parallel to AB. Then AD. DE : : *b* AF.FG, and DE. EC *b* : : DI. IH : : *c* FG. GB *Which was to be done.* b 2. 6. c 34. 1. & 7. 5.

Schol.



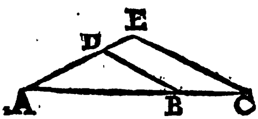
Hence we learn to cut a right line given AB, *into as many equal parts as we please* (*suppose* 5;) which will be more easily performed thus.

Draw an infinite line AD, and another BH parallel to it, and infinite also. Of these take equal parts, AR, RS, SV, VN; and BZ, ZX, XT, TL; in each line less parts by one, than are required in AB; then let the right lines LR, TS, XV, ZN, be drawn; these lines so drawn shall cut the right line given AB into five equal parts.

For

a 33. 1. For RL, ST, VX, NZ, are *a* parallels; therefore,
b *constr.* whereas AR, RS, SV, VN are *b* equal; *c* thence AM,
c 2. 6. MO, OP, PQ, are equal also. Likewise, because that BZ = ZX, therefore is BQ = PQ, and therefore AB is cut into five equal parts. *Which was to be done.*

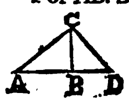
PROP. XI.



Two right lines being given AB, AD, *to find out a third in proportion to them* (DE)

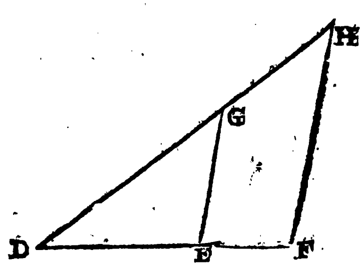
Join BD, and from AB being produced take BC = AD. Through C draw CE parallel to BD; with which let AD produced meet in E, then is DE the proportional required.

a 2. 6. For AB. BC (AD) *a* : : AD. DE. *Which was to be done.*



Or thus: make the angle ABC right, and also the angle ACD right, then
b 1 *cor.*8.6. *b* AB. BC : : BC. BD.

PROP. XII.

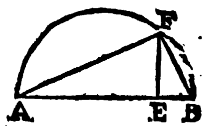


Three right lines being given, DE, EF, DG, *to find out a fourth proportional* GH.

Join EG, and through F draw FH parallel to EG; with which let DG produced to H meet. Then it is
a 2. 6. evident that DE. EF *a* : : DG. GH. *Which was to be done.*

PROP.

PROP. XIII.



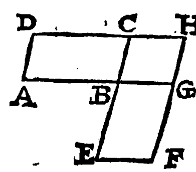
Two right lines being given AE, EB, *to find a mean proportional* EF.

Upon the whole line AB as a diameter, describe a semicircle AFB, and from E erect a perpendicular FE meeting with the periphery in F, then AE. EF : : EF. EB. For let AF and FB be drawn; *a* then from the right angle of the right angled triangle AFB is drawn a right line FE perpendicular to the base. *b* Therefore AE. FE : : FE. EB. *Which was to be done.* a 31. 3. b *cor.* 8 6.

Coroll.

Hence, a right line drawn in a circle from any point of a diameter, perpendicular to that diameter, and produced to the circumference, is a mean proportional betwixt the two segments of that diameter.

PROP. XIV.



Equal Parallelograms BD, BF, *having one angle* ABC, *equal to one* EBG, *have the sides which are about the equal angles reciprocal* (AB. BG : : EB. BC;) *and those parallelograms* BD, BF, *which have one angle* ABC *equal to one* EBG, *and the sides which are about the equal angles reciprocal, are equal*

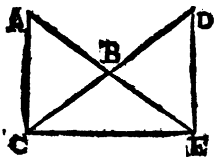
For let the sides AB, BG, about the equal angles make one right line; *a* wherefore EB, BC, shall do the same. Let FG, DC, be produced till they meet. a *sch.* 15.1.

1. *Hyp.* AB. BG *b* : : BD. BH : : *c* BF. BH : : *d* BE. BC, *e* therefore, &c. b 1. 6. c 7. 5.

2. *Hyp* BD. BH : : *f* AB. BG : : *g* BE. BC : : *h* BF. BH. *k* Therefore the Pgr. BD = BF. *Which was to be demonstrated.* d 1. 6. e 11. 5. f 1. 6. g *hyp.* h 1. 6. k 11. *and* 9. 5.

PROP.

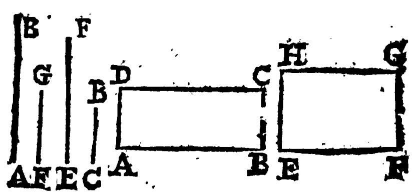
PROP. XV.



Equal triangles having one angle ABC, *equal to one* DBE, *their sides which are about the equal angles are reciprocal* (AB. BE : : DB. BC.) *And those triangles that have one angle* ABC *equal to one* DBE, *and have also the sides that are about the equal angles reciprocal* (AB. BE : : DB. BC) *are equal.*

Let the sides CB, BD, which are about the equal an-
a sch. 15. 1. gles be set in a strait line; *a* therefore ABE is a right
b 1. 6. line. Let CE be drawn.
c 7. 5. 1. *Hyp.* AB. BE : : *b* the triangle ABC. CBE *c* : : the
d 1. 6. triangle DBE. CBE : : *d* DB. BC, *e* therefore &c.
e 11 5. 2. *Hyp.* The triangle ABC. CBE : : *f* AB. BE : : *g* DB.
f 1. 6. BC *h* : : the triangle DBE CBE. *k* Therefore the tri-
g *hyp.* angle ABC = DBE. *Which was to be demonstrated.*
h 1. 6.
k 11. *and* 9 5.

PROP. XVI.



If four right lines are proportional (AB. FG : : EF. CB) *the rectangle* AC *comprehended under the extremes* AB, CB, *is equal to the rectangle* EG *comprehended under the means* FG, EF. *And if the rectangle* AC *comprehended under the extremes* AB, CB, *be equal to the rectangle* EG, *comprehended under the means* FG, EF, *then are the four right lines proportional.* (AB FG : : EF. CB)

a 12. *ax.* 1 *Hyp.* The angles B and F are right, and *a* conse-
quently equal, and by hypothesis AB. FG : : EF. CB;
b 14. 6. *b* therefore the rectangle AC = EG.

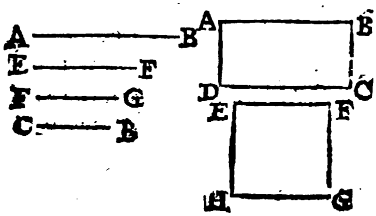
2. *Hyp.*

2. *Hyp.* The rectangle AC *c* = EG, and the angle B = F; *d* therefore AB. FG : : EF. CB. *Which was to be demonstrated.* c *hyp.* d 14. 6.

Coroll.

Hence it is easy to apply a rectangle given EG to a right-line given AB; (*viz.*) *e* by making AB. EF : : FG. BC. e 12. 6.

PROP. XVII.



If three right lines are proportional AB EF : : EF. CB) *the rectangle* AC *made under the extremes* AB, CB *is equal to the square* EG *made of the middle* EF. *And if the rectangle* AC, *comprehended under the extremes* AB, CB, *be equal to the square* EG, *made of the middle* EF, *then the three lines are proportional,* (AB. EF : : EF. CB)

Take FG = EF.

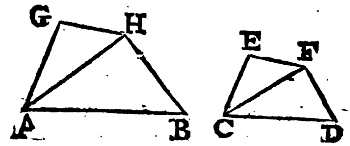
1. *Hyp.* AB. EF : : *a* EF. (FG.) CB, therefore the rectangle AC *b* = EG *c* = EFq. a *hyp.* b 16. 6.

2. *Hyp.* The rectangle AC *d* = to the square EG = EFq. *e* therefore AB. EF : : FG (EF.) BC. *Which was to be demonstrated.* c 29. *def.* 1. d *hyp.* e 16. 6.

Coroll.

Let A x B = Cq, therefore A. C : : C. B.

PROP. XVIII.

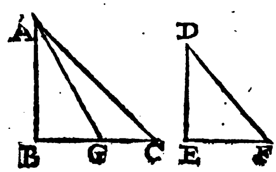


Upon a right line given AB, *to deſcribe a right-lined figure* AGHB, *like and alike ſituate to a right-lined figure given* CEFD.

a 23. 1. Reſolve the right-lined figure given into triangles; *a* Make the angle ABH = D, *a* and the angle BAH = DCF, *a* and the angle AHG = CFE, *a* and the angle HAG = FCE, then AGHB ſhall be the right-lined figure ſought.

b *conſtr.* For the angle B *b* = D, and the angle BAH *b* = DCF,
c 32. 1. *c* wherefore the angle AHB = CFD, *b* alſo the angle HAG = FCE, and the angle AHG *b* = CFE, *c* wherefore
d 2. *ax.* the angle G = E, and the whole angle GAB *d* = ECD, and the whole angle GHB *d* = EFD. The Polygons therefore are mutually equiangular. Moreover becauſe
e 4. 6. the triangles are equiangular, therefore AB. BH *e* :: CD. DF; and AG. GH *e* :: CE. EF. Likewiſe AG. AH
f 22. 5 :: *e* CE. CF, and AH. AB :: CF. CD. *f* From whence by equality AG. AB :: CE. CD. After the ſame man-
g 1 *def.* 6. ner GH. HB :: EF. FD. *g* Therefore the Polygons ABHG, CDFE are like and alike ſituate. *Which was to be done.*

PROP. XIX.



Like triangles ABC, DEF, *are in duplicate ratio of their homologous ſides,* BC, EF.

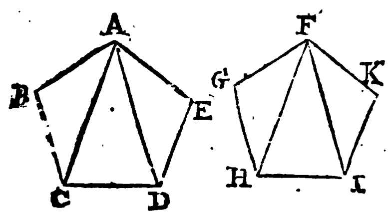
a 11. 6. *a* Let there be made BC. EF :: EF. BG, and let AG be drawn. Be-
b *cor.* 4. 6 cauſe that AB. DE *b* :: BC. EF *c* :: EF. BG, and the
c *conſtr.* angle B = E, *d* therefore is the triangle AEG = DEF.
d 15. 6. But the triangle ABC. ABG :: *e* BC. BG, and *f*
e 1. 6 BC
f 10. *def.* 5.

$\frac{BC}{BG} = \frac{BC}{EF}$ twice; therefore $\frac{ABC}{ABG}$ that is, $\frac{ABC}{DEF}$ g $= \frac{BC}{EF}$ twice. *Which was to be demonstrated.* g 11. 5.

Coroll

Hence, If three right-lines (BC, EF, BG) are proportional, then as the first is to the third, so is a triangle made upon the first BC, to a triangle like and alike described upon the second EF; or so is a triangle described upon the second EF, to a triangle like and alike described upon the third.

PROP. XX.



Like Polygons ABCDE, FGHIK, *are divided into equal triangles* ABC, FGH, *and* ACD, FHI, *and* ADE, FIK; *both equal in number and homologous to the wholes* (ABC. FGH : : ABCDE. FGHIK : : ACD. FHI : : ADE. FIK.) *And the Polygons* ABCDE, FGHIK, *have a duplicate ratio one to the other of what one homologous side* BC *hath to the other homologous side* GH.

1. For the angle B *a*=G, and AB. BC *a* : : FG. GH. *b* Therefore the triangles ABC, FGH, are equiangular. After the same manner are the triangles AED, FKI like. Since therefore the angle BCA *b* = GHF, and the angle ADE *b*= FIK, and the whole angles BCD, GHI, and the whole angles CDE, HIK are *c* equal, there remains the angle ACD *d* = FHI, and the angle ADC=FIH; *e* from whence also the angle CAD = HFI, therefore the triangles ACD, FHI are like. Therefore, &c.

a *hyp.* b 6. 6. b 6 6. c *hyp.* d 3. *ax.* e 32. 1. f 19 6.

2. Because the triangles BCA, GHF are like, *f* is $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the same reason is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; lastly $\frac{DEA}{IKF} = \frac{DE}{IK}$ twice. Now where

g *hyp.* & 16. 5. as that BC. GH g :: CD. HI g :: DE. IK, h therefore is the triangle BCA. GHF :: CAD. HFI :: DEA. IKF k

h *cor.* 23 5 k 12. 5. :: the polygon ABCDE. FGHIK :: $\frac{BC}{GH}$ twice.

Coroll.

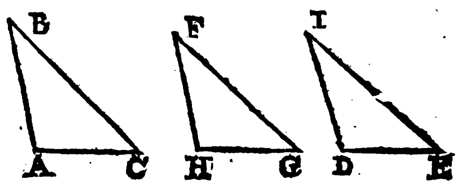
1. Hence if there are three right-lines proportional, then as the first is to the third, so is a polygon made upon the first to a polygon made on the second like and alike described; or so is a polygon made upon the second, to a polygon made on the third like and alike described.

Hence we have a method of inlarging or diminishing any right-lined figure in a ratio given: For if you would make a pentagon quintuple of that pentagon whereof CD is the side, then betwixt AB and 5 AB find out a

* 18. 6. mean proportional, * upon this raise a pentagon like to that given, and it shall be quintuple of the pentagon given.

2. Hence also, If the homologous sides of like figures be known, then will the proportion of the figures be evident, *viz.* by finding out a third proportional.

PROP. XXI.

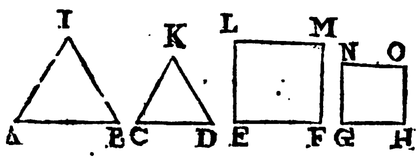


Right lined figures ABC, DIE, *which are like to the same right-lined figure* HFG, *are also like one to the other.*

a 1. *def.* 6. For the angle A a= H a= D; and the angle C a= G a= E; and the angle B a= F a= I. Also a AB AC :: HF. HG :: DI. DE, and a AC. CB :: HG. GF :: DE. EI. And AB. BC :: HF. FG :: DI. IE. Therefore a ABC, DIE, are like. *Which was to be demonstrated.*

PROP.

PROP. XXII.



If four right-lines are proportional (AB. CD : : EF. GH) *the right-lined figures also described upon them being like and in like sort situate, shall be proportional* (ABI. CDK : : EM. GO) *And if the right-lined figures described upon the lines, like and alike situate, be proportional* (ABI. CDK : : EM. GO) *then the right lines also shall be proportional* (AB. CD : : EF. GH)

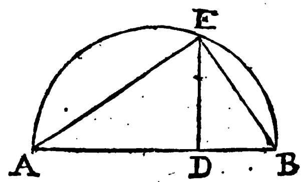
1. *Hyp.* $\frac{ABI}{CDK}$ *a*= $\frac{AB}{CD}$ twice = *a* $\frac{EF}{GH}$ twice *a*= $\frac{EM}{GO}$ a 19. 6.
b therefore ABI. CDK : : EM. GO.

2. *Hyp.* $\frac{AB}{CD}$ twice *a*= $\frac{ABI}{CDK}$ *b*= $\frac{EM}{GO}$ = *c* $\frac{EF}{GH.}$ b *hyp.*
twice Therefore *d* AB. CD : : EF. GH. *Which was* c 20 6.
to be demonstrated. d *cor.* 23. 5.

Schol.

Hence is deduced the manner and reason of multiplying surd quantities, ex. g. Let $\sqrt{5}$ be to be multiplied into $\sqrt{3}$. I say that the product will be $\sqrt{15}$. For by the definition of multiplication it ought to be, as 1. $\sqrt{3}$: : $\sqrt{5}$, to the product. Therefore by *this* q. 1. q. $\sqrt{3}$: : q. $\sqrt{5}$. q. of the product. That is, 1. 3 : : 5, to the square of the product, therefore the square of the product is 15. Wherefore $\sqrt{15}$ is the product of $\sqrt{3}$ into $\sqrt{5}$. *Which was to be demonstrated.*

THEOREM.



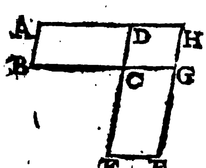
Pet, Herig. *If a right-line* AB *be cut any-wise in* D, *the rectangle comprehended under the parts* AD, DB, *is a mean proportional betwixt their squares. Likewise the rectangle comprehended under the whole* AB, *and one part* AD, *or* D B *is a mean proportional betwixt the square of the whole* AB, *and the square of the said part*, AD, *or* DB.

Upon the diameter AB describe a semicircle; from D erect a perpendicular DE, meeting with the periphery in E, join AE, BE.

a cor. 8 6. It's evident that AD. DE *a* : : DE. DB, *b* therefore
b 22. 6. ADq. DEq : : DEq DBq, *c* that is, ADq. ADB : : ADB.
c 17. 6. DBq. *Which was to be demonstrated.*
d cor 8. 6. Moreover BA. AE : : *d* AE AD, *e* therefore BAq.
e 22. 6. AEq : : AEq. ADq. *f* that is, BAq. BAD : : BAD. ADq.
f 17. 6. After the same manner ABq. ABD : : ABD. BDq. *Which was to be demonstrated*

Or thus: suppose Z=A+E. It is manifest that Aq.
a 1. 6. AE : : *a* A. E : : *a* AE. Eq. also Zq. ZA : : *a* Z. A : : ZA. Aq, and Zq. ZE : : *a* Z. E : : ZE. Eq.

PROP. XXIII.



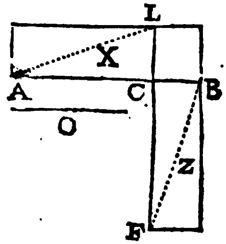
Equiangular parallelograms AC, CF, *have the ratio one to the other, which is compounded of their sides.* $\left(\frac{AC}{CF}=\frac{BC}{CG}+\frac{DC}{CE}\right)$

Let the sides about the equal,
a sch. 15, angles C be *a* set in a direct line, and let the Pgr. CH be compleated. Then is the ratio of
b 20. def. 5. c 1. 6. $\frac{AC}{CF} \,b\!= \frac{AC}{CH}+\frac{CH}{CF}\,c\!=\frac{BC}{CG}+\frac{DC}{CE}$, *Which was to be demonstrated.*

Coroll.

Coroll.

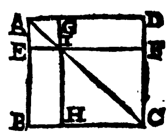
Hence, *and from* 34. 1. *it appears*, 1. *That triangles which have one angle equal* (*as at* C) *have a ratio compounded of the ratio's of the right-lines*, AC *to* CB, *and* LC *to* CF,) *containing the equal angle.* *Andr. Tacq.* 15. 5



2. *That all rectangles, and* * *consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude.* After the like manner you may argue in triangles * 35. 1.

3. *From hence is apparent how to give the proportion of triangles, and parallelograms.* Let there be two Pgrs. X and Z, whose bases are AC, CB, and altitudes CL, CF. Make CL. CF : : CB. O, * then will it be X. Z : : AC. O. * 14. 6. *and* 1. 6.

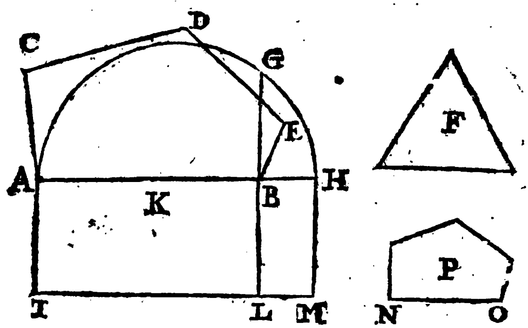
PROP. XXIV.



In every parallelogram ABCD, *the parallelograms* EG, HF *which are about the diameter* AC, *are like to the whole, and also one to the other.*

For the Pgrs. EG, HF, have each of them one angle common with the whole; *a* therefore they are equiangular to the whole, and also one to the other Also both the triangles ABC, AEI, IHC *a* and the triangles ADC, AGI, IFC are equiangular mutually; *b* therefore AE. EI : : AB. BC, and *b* AE. AI : AB. AC, and *b* AI. AG : : AC AD. *c* Therefore by equality, AE. AG : : AB AD. *d*. Therefore the Pgrs EG, BD are like. After the same manner are HF, BD like also. Therefore, &c. a 29. 1. b 4. 6. c 22. 5. d 1 def. 6.

PROP. XXV.

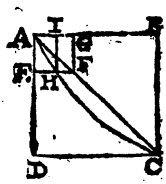


Unto the right-lined figure given ABEDC, *to describe another figure* P, *like and alike situate, which also shall be equal to another right-lined figure given* F.

a 45. 1. *a* Make the rectangle AL = ABEDC; *b* also upon
b 44. 1. BL make the rectangle BM = F; betwixt AB and BH
c 13. 6. *c* find out a mean proportional NO; Upon NO *d* make
d 18. 6. the polygon P like to the right-lined figure given ABEDC. I say the polygon P so made, shall be equal to F, that was given.

e *cor* 20. 6. For ABEDC (AL.) P :: AB. BH :: *f* AL. BM. There-
f 1. 6. fore P *g* = BM *h* = F. *Which was to be done.*
g 14. 5.
h *constr.*

PROP. XXVI.

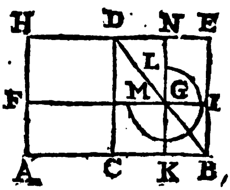


If from the parallelogram ABCD, *be taken away another parallelogram* AGFE, *like unto the whole, and in like sort set, having also an angle* EAG *common with it; then is that parallelogram about the same diagonal* AC *with the whole.*

If you deny AC to be the common diagonal, then let AHC be it, cutting EF in H, and let HI be drawn pa-
a 24. 6. rallel to AE. Then are Pgrs. EI, DB, *a* like, *b* there-
b 1. *def.* 6. fore AE. EH :: AD. DC :: *c* AE. EF, and *d* consequent-
c *hyp.* ly EH = EF. *f Which is absurd.*
d 9. 5.
f 9. *ax.*

PROP.

PROP. XXVII.

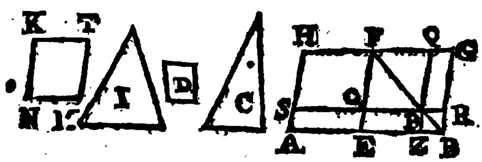


Of all parallelograms AD, AG *applied to the same right line* AB, *and wanting in figure by the parallelograms* CE, KI, *like and alike set to the Pgr.* AD, *which is described upon the half line, the greatest is that* AD, *which is applied to the half line being like to the defect* KI.

For because that GE *a* = GC, if KI which is common be added, *b* then is KE = CI *c* = AM, add CG which is common, *d* then is AG = to the Gnomon MBL; *e* but the Gnomon MBL ⊐ CE (AD.) Therefore AG ⊐ AD. *Which was to be demonstrated.*

a 43. 1.
b 2. *ax.*
c 36. 1.
d 2. *ax.*
e 9 *ax.*

PROP. XXVIII.



To a right-line given AB, *to apply a parallelogram* AP, *equal to a right-lined figure given* C, *deficient by a parallelogram* ZR, *which is like to another parallelogram given* D; * *but it is necessary that the right-lined figure given* C, *to which the Pgr. to be applied* AP *must be equal, be not greater than the Pgr.* AF *which is applied to half the line, since the defects both of* AF, *which is apply'd to half the line, and of* AP *the parallelogram to be applied, must be like.* * 27. 6.

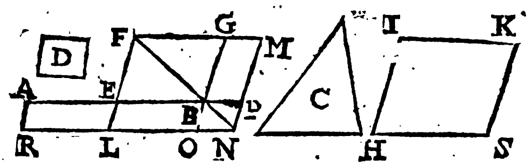
Bisect AB in E; upon EB *a* make the Pgr. EG like to the Pgr. D; and *b* let EG = C + I. *c* Make the Pgr. NT = I, and like to the Pgr. given D, or EG; draw the diameter FB; Make FO = KN, and FQ = KT; thro' O and Q draw the parallels SR, QZ. Then is the Pgr. AP that which was sought.

a 18 6.
b *sch.* 45. 1.
c 25. 6.

For

d constr. & 24. 6 For the Pgrs. D, EG, OQ, NT, ZR, are all *d* like one to the other, and the Pgr. EG = *e* NT + C = *e* OQ
e constr. + C, *f* wherefore C = to the Gnomon OBQ *g* = AO +
f 3. ax. PG = *h* AO + EP = AP. *Which was to be done.*
g 2. ax.
h 43. 1.

PROP. XXIX.

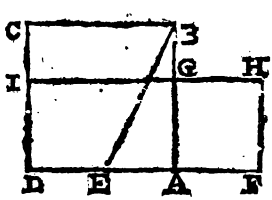


Upon a right-line given AB, *to apply a parallelogram* AN *equal to a right-lined figure given* C, *exceeding by a Pgr.* OP, *which shall be like to another Pgr. given* D.

a 18. 6. Bisect AB in E. Upon EB *a* make a Pgr. EG like to
b 25. 6. the given one D, and *b* let the Pgr. HK = EG + C, and
c 3. 1. like to the given one D, or to EG. Make FEL = *c* IH; and *c* FGM = IK. Thro' L, M, draw the parallels MN and RN; and AR parallel to NM. Produce ABP, GBQ; draw the diameter FBN. Then is AN the parallelogram required.

d constr. For the Pgrs. D, HK, LM, EG, are *d* like, *e* therefore
e 24. 6. the Pgr. OP is like to the Pgr. LM, or D. Also LM *f*
f constr. = HK *f* = EG + C. *g* Therefore C = to the Gnomon
g 3. ax. ENG. But AL *h* = LB *k* = BM; *l* therefore C = AN.
h 36. 1. *Which was to be done.*
k 43. 1.
l 2. and 1. ax.

PROP. XXX.

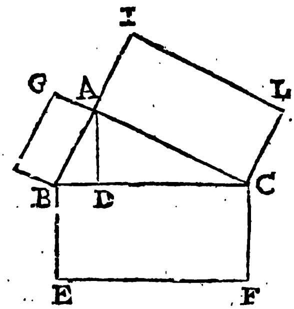


To cut a finite right-line given AB, *according to extreme and mean ratio* (AB. AG : : AG. GB.)

a 11. 2. *a* Cut AB in G, in such wise that AB x BG = AGq.
b 17. 6. *b* Then BA. AG : : AG. GB. *Which was to be done.*

PROP.

PROP. XXXI.



In right-angled triangles BAC, *any figure* BF *described upon the side* BC *subtending the right angle* BAC, *is equal to the figures* BG, AL, *which are like and alike situate to the former* BF, *and described upon the sides* BA, AC, *containing the right angle.*

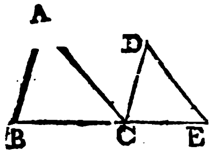
From the right-angle BAC let fall the perpendicular AD. Because DC. CA : : *a* CA. CB, *b* therefore AL. BF : : DC. GB. Also, because DB. BA : : *a* BA. BC, *b* therefore BG. BF : : DB. BC ; *c* therefore AL+BG. BF : : DC+DB (BC) BC. *d* Therefore AL+BG= BF. *Which was to be demonstrated.* a *cor.* 8. 6. b *cor.*20. 6. c 24. 5. d *sch.* 14. 5.

Or thus : BG. BF : : *e* BAq. BCq. And *e* AL. BF : : ACq. BCq, *f* therefore BG+AL. BF : : BAq+ACq. BCq. *g* Therefore whereas BAq+ACq=*h* BCq ; *h* thence is BG+AL=BF. *Which was to be demonstrated* e 22. 6. f 24. 5. g *sch.* 14. 5. h 47. 1.

Coroll.

From this proposition you may learn how to add or subtract any like figures, by the same method that is used in adding and subtracting of squares, in *Schol.* 47. 1.

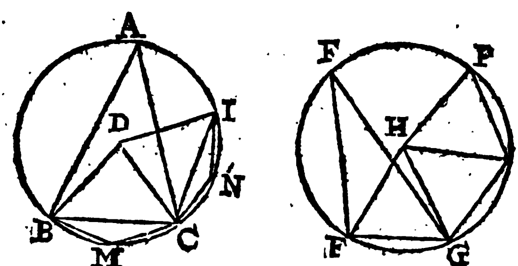
PROP. XXXII.



If two triangles ABC, DCE *having two sides proportional to two* (AB. AC : : DC. DE.) *be so compounded or set together at one angle* ACD, *that their homologous sides are also parallel* (AB *to* DC, *and* AC *to* DE,) *then the remaining sides of those triangles* (BC, CE) *shall be found placed in one strait line.*

a 29. 1. For the angle A *a* = ACD *a* = D, and AB. AC *b* : : DC. DE, *c* therefore the angle B = DCE. Therefore
b *hyp.* the angle B + A *d* = ACE; but the angle B + A + ACB *e*
c 6. 6. = 2 right, *f* therefore the angle ACE + ACB = 2 right;
d 2. *ax.* *g* therefore BCE is a right-line. *Which was to be de-*
e 32. 1. *monstrated.*
f 1. *ax.*
g 14. 1.

PROP. XXXIII.



In equal circles DBCA, HFGP, *the angles* BDC, FHG, *have the same ratio with the peripheries* BC, FG, *on which they insist; whether the angles be set at the centers (as* BDC, FHG) *or at the circumferences,* A, E: *And so likewise have the Sectors* BDC, FHG.

Draw the right-lines BC, FG. Make CI = CB, and GL = FG = LP, and join DI, HL, HP.

a 28. 3. The arch BC *a* = CI, *a* also the arches FG, GL, LP,
b 27. 3. are equal; *b* therefore the angle BDC = CDI, *b* and the angle FHG = GHL = LHP. Therefore the arch BI is the same multiple of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch FP, the same multiple of the arch FG, as the angle FHP is of the angle FHG. But if the arch BI $\sqsubset$, =, $\sqsupset$ FP,
c 27. 3. *c* then likewise is the angle BDI $\sqsubset$, =, $\sqsupset$ FHP.

There-

Therefore is the arch BC. FG :: *d* the angle BDC. FHG d 6 *def*. 5.

e :: $\frac{BDC}{2}$. $\frac{FHG}{2}$ *f* :: A. E. *Which was to be demon.* e 15. 5.

Moreover, the angle BMC *g* = CNI; *h* and there- f 20. 3.
fore the segment BCM = CIN. *k* Also the triangle g 27. 3.
BDC = CDI; *l* wherefore the sector BDCM = CDIN. h 24. 3.
After the same manner are the sectors FHG, GHL, k 4. 1.
LHP equal one to other. Therefore since accordingly l 2. *ax*.
as the arch BI $\sqsubset$, $=$, $\sqsupset$ FGP, so is likewise the
sector BDI $\sqsubset$, $=$, $\sqsupset$ FHP; *m* thence shall be the m 6. *def*. 5.
sector BDC. FHG :: the arch BC. FG. *Which was to be demonstrated.*

Coroll.

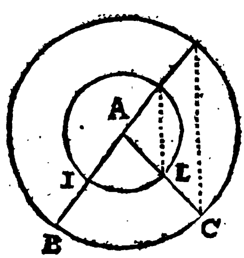
1. Hence, *As sector is to sector, so is angle to angle.* 11. 5.

2. *The angle* BDC *in the center, is to four right angles, as the arch* BC, *on which it insists, to the whole circumference.*

For as the angle BDC is to a right-angle, so is the arch BC to a quadrant. Therefore BDC is to four right-angles, as the arch BC is to four quadrants, that is, to the whole circumference. Also, the angle A. 2 right :: the arch BC. periphery.

3. Hence, *the arches* IL, BC, *of unequal circles which subtend equal angles, whether at the centers, as* IAL *and* BAC, *or at the periphery, are like.*

For IL. periph. :: angle IAL (BAC) 4 right. Also, Arch BC. periph :: angle BAC. 4 right. Therefore IL. periph :: BC. periph. And consequently the arches IL and BC are like. Whence



4. *Two semidiameters* AB, AC, *cut off like arches* IL, BC *from concentrick peripheries.*

The End of the sixth Book.

THE

The Seventh Book of *EUCLIDE's* Elements.

Definitions.

I. UNity is that, by which every thing that is, is called One.

II. Number is a multitude composed of units.

III. Part is a number of a number, the lesser of the greater, when the lesser measureth the greater.

Every part is denominated from that number, by which it measures the number whereof it is a part; as 4 *is called the third part of* 12, *because it measures* 12 *by* 3.

IV. But when the lesser number does not measure the greater, then the lesser is call'd, not a part, but parts of the greater.

All parts whatsoever are denominated from those two numbers, by which the greatest common measure of the two numbers measures each of them; as 10 *is said to be two thirds of the number* 15; *because the greatest common measure, which is* 5, *measures* 10 *by* 2, *and* 15 *by* 3.

V. A multiple is a greater number compared with a lesser, when the lesser measures the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or that which differeth from an even number by unity.

VIII. A number evenly even, is that which an even number measureth by an even number.

IX. But a number evenly odd, is that which an even number measureth by an odd number.

X. A number oddly odd, is that which an odd number measureth by an odd number.

XI. A prime (or first) number is that which is measured only by unity.

XII. Numbers prime the one to the other, are such as only unity doth measure, being their common measure.

XIII. A composed number is that which some certain number measureth.

XIV. Numbers composed the one to the other, are those, which some number, being a common measure to them both, doth measure.

In this, and the preceding definition, unity is not a number.

XV. One number is said to multiply another when the number multiplied is so often added to it self, as there are units in the number multiplying, and another number is produced.

Hence in every multiplication unity is to the multiplier, as the multiplicand is to the product.

Obs. *That many times, when any numbers are to be multiplied (as* A *into* B) *the conjunction of the letters denotes the product: So* AB = A x B, and CDE = C x D x E.

XVI. When two numbers multiplying themselves produce another, the number produced is called a plain number; and the numbers which multiplied one another, are called the sides of it: *So* 2 (C) x 3 (D) = 6 = CD *is a plane number.*

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a solid number; and the numbers multiplying one another, are called the sides thereof: *So* 2 (C) x 3 (D) x 5 (E) = 30 = CDE *is a solid number.*

XVIII. A square number is that which is equally equal; or, which is contained under two equal numbers. *Let* A *be the side of a square; the square is thus noted,* AA, *or* Aq,

XIX. A Cube is that number which is equally equal equally; or, which is contained under three equal numbers. *Let* A *be the side of a Cube; the Cube is thus noted,* AAA, *or* Ac.

In this definition, and the three foregoing, unity is number.

XX. Numbers are proportional, when the first is the same multiple of the second, as the third is of the fourth;

fourth; or, the same part; or, when a part of the first number measures the second, and the same part of the third measures the fourth, equally: and *vice versa. So* A. B. :: C. D, *that is*, 3. 9 :: 5. 15.

XXI. Like plane, and solid numbers, are those which have their sides proportional: *Namely, not all the sides, but some.*

XXII. A perfect Number is that which is equal to its own parts.

As 6, *and* 28. *But a number that is less than it's parts is called an* Abounding *number; and one which is greater, a* Diminutive: *so* 12 *is an abounding*, 15 *a diminutive number.*

XXIII. One number is said to measure another, by a third number, which when it either multiplies, or is multiplied by the measuring number, produces the number measured.

In Division, unity is to the quotient, as the divisor is to the dividend. Note, *that a number placed under another with a line between them, signifies division*: So $\frac{A}{B}$ = A divided by B, and $\frac{CA}{B}$ = C x A divided by B.

Those two numbers are called the Terms or Roots of a Proportion, than which lesser cannot be found in the same proprtion.

Postulates, or Petitions.

1. That numbers equal or multiple to any number may be taken at pleasure.

2. That a number greater than any other whatsoever may be taken.

3. That Addition, Subtraction, Multiplication, Division, and the Extractions of Roots or sides of square and cube numbers be also granted as possible.

Axioms.

1. Whatsoever agrees with one of many equal numbers, agrees likewise with the rest.

2. Those parts that are the same to the same part, or parts, are the same among themselves.

3. Numbers that are the same parts of equal numbers, or of the same number, are equal among themselves.

4. Those

4. Those numbers, of which the same number or equal numbers, are the same parts, are equal amongst themselves.

5. Unity measures every number by the units that are in it; that is, by the same number.

6. Every number measures it self by unity.

7. If one number multiplying another, produces a third, the multiplier shall measure the product by the multiplied; and the multiplied shall measure the same by the multiplier.

Hence, No prime number is either a plane, solid, square, or cube number.

8. If one number measures another, that number by which it measureth shall measure the same by the units that are in the number measuring, that is, by the number it self that measures.

9. If a number measuring another, multiply that by which it measureth, or be multiplied by it, it produceth the number which it measureth.

10 How many numbers soever any number measureth, it likewise measureth the numbers composed of them.

11. If a number measures any number, it also measureth every number which the said number mearsureth.

12. A number that measures the whole and a part taken away, doth also measure the residue.

PROP. I.

A.....E. G.B 8 5 3
C . F..D $\frac{5}{3}$ $\frac{3}{2}$ $\frac{2}{1}$
H---

Two unequal numbers AB, CD, *being given, if the lesser* CD, *be continually taken from the greater* AB (*and the residue* EB *from* CD, &c.) *by an alternate subtraction, and the number remaining do never measure the precedent, till unity* GB *be taken; then are the numbers which were given* AB, CD, *prime the one to the other.*

If you deny it, let AB, CD, have a common measure, namely the number H, therefore H measuring CD, doth *a* also measure AE; and *b* consequently the remainder EB; *a* therefore it likewise measures CF, and *b* so the remainder FD; *a* therefore it also measures EG. But it measured the whole EB, and *b* therefore it must measure that which remaineth GB, that is, a number measures unity. *c Which is absurd*

a 11. *ax*. 7.
b 12. *ax*. 7.
c 9. *ax*. 1.

PROP. II.

Two numbers AB, CD *being given, not prime the one to the other, to find out their greatest common measure* FD.

```
    9      6
A.........E......B 15 9 6
   6    3
C......F...D  3/3 6/3 3/3
           G---
```

Take the lesser number CD from the greater AB as
a 6 *ax.* 7. often as you can. If nothing remains, *a* it is manifest that CD is the greatest common measure. But if there remains something (as EB) then take it out of CD, and the residue FD out of EB, and so forward till some number
b 1. 7. (FD) measure the said EB (*b* for this will be, before you come to unity) FD shall be the greatest common measure.

c *constr.* For FD *c* measures EB, and *d* therefore also CF; and
d 11. *ax.* 7. *e* consequently the whole CD; *d* therefore likewise
e 12. *ax.* 7. AE; and so measures the whole AB. Wherefore it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G) then where-
d 11 *ax* 7 as G measureth CD, it *d* must likewise measure AE, *e*
e 12. *ax.* 7. and the residue EB, *d* as also CF, *e* and by consequence
g *suppos.* the residue FD, *g* the greater the less. *Which is absurd.*
h 9. *ax.* 1.

Coroll.

Hence, A number that measures two numbers, does also measure their greatest common measure.

PROP. III.

```
A..........12
B........8
     D....4
C.....6
     E..2
F---
```

Three numbers being given, A, B, C, *not prime one to another, to find out their greatest common measure* E.

Find out D the greatest common measure of the two numbers A, B. If D measures C the third, it is clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll. of the Proposition preceding. Therefore let E be the greatest common measure of the said numbers D and C, and it shall be the number which was required.

For

For E *a* meaſures C and D, and D meaſures A and B; therefore *b* E meaſures each of the numbers A, B, C: neither ſhall any greater (F) meaſure them; for if you affirm that, *c* then F meaſuring A and B, does likewiſe meaſure D their greateſt common meaſure; and in like manner, F meaſuring D and C, does alſo meaſure E *c* their greateſt common meaſure, *d* the greater the leſs. *e Which is abſurd.*

a *conſt.* b 11. *ax.* 7. c *cor.* 1. 7. d *ſuppoſ.* e 9. *ax.* 1.

Coroll.

Hence, a number that meaſures three numbers, does alſo meaſure their greateſt common meaſure.

PROP. IV.

A.......6
B.......7
B..................18
B.........9

Every leſs number A *is of every greater* B *either a part or parts.*

If A and B be prime to one another, *a* A ſhall be as many parts of the number B, as there are units in A (as 6 = $\frac{6}{7}$ of 7). But if A meaſures B, it is *b* plain that A is a part of B (as 6 = $\frac{1}{3}$ of 18.) Laſtly, if A and B be otherwiſe compoſed to one another, *c* the greateſt common meaſure ſhall determine how many parts A does contain of B; as 6 = $\frac{2}{3}$ of 9.

a 4. *def.* 7. b 3. *def.* 7. c 4. *def.* 7.

PROP. V.

A.......6 D.....4
6 6 4 4
B......G......C 12 E.....H.....F 8.

If a number A *be a part of a number* BC, *and another number* D *the ſame part of another number* EF; *then both the numbers together* (A+D) *ſhall be the ſame part of both the numbers together* (BC+EF,) *which one number* A *is of one number* BC.

For if BC be reſolved into its parts BG, GC, equal to A; and EF alſo into its parts EH, HF, equal to D; *a* the number of parts in BC ſhall be equal to the number of parts in EF. Therefore ſince A+D *b* = BG+EH = GC+HF, thence A+D ſhall be as often in BC+EF, as A is in BC. *Which was to be demonſtrated.*

a *hyp.* b *conſt.* & 2. *ax.* 1.

c 2. ax. 1. Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$, then $2a = x$, and $2b = y$, therefore $2a + 2b = x + y$, therefore $a + b = \frac{x+y}{2}$. *Which was to be demonstrated.*

PROP. VI.

A^3...G^3...B 6 D....H^4....E^4 8
C ....9 F........... 12

If a number AB *be parts of a number* C, *and another number* DE *the same parts of another number* F; *then both numbers together* AB+DE *shall be of both numbers together* C + F *the same parts, that one number* AB *is of one number* C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in both AB, DE,
a hyp. is equal by supposition; since then AG *a* is the same part of the number C, that DH is of the number F, AG +
b 5. 7. DH *b* shall be the same part of the compounded number C+F, that one number AG is of one number C. *b* In like manner GB+HE is the same part of the said C+F, that one number GB is of one number C.
c 2. ax. 7. *c* Therefore AB + DE is the same parts of C + F, that AB is of C. *Which was to be demonstrated*

Or thus. Let $a = \frac{2}{3}x$, and $b = \frac{2}{3}y$, and $x + y = g$, then, because $3a = 2x$, and $3b = 2y$, is $3a + 3b = 2x + 2y = 2g$, therefore $a + b = \frac{2}{3}g = \frac{2}{3} : x + y$.

PROP. VII.

A^5.....E^3...B 8
G^6......C^{10}..........F^6......D 16

If a number AB *be the same part of a number* CD, *that a part taken away* AE *is of a part taken away* CF; *then shall the residue* EB *be the same part of the residue* FD *that the whole* AB *is of the whole* CD.

a 1 post. 7. *a* Let EB be the same part of the number GC that
b 5. 7. AB is of CD, or AE of CF, *b* therefore AE+EB is the same part of CF+GC that AE is of CF, or AB
c 6. ax. 1. of CD, *c* therefore GF=CD. Take away CF com-
d 3. ax. 1. mon to both, and *d* there remains GC=FD, *e* Where-
e 2. ax. 7. fore

fore EB is the same part of the residue FD (GC) that the whole AB is of the whole CB. *Which was to be demonstrated.*

Or thus. Let $a+b=x$; and $c+d=y$; and $x=3y$, in like manner as $a=3c$; I say $b=3d$. For $3c+3d$ *f* $=3y=x$ *g* $=a+b$, take away from both $3c$ *g* $=a$, and there remains $3d=b$. *Which was to be demonstrated.* f 1. 2. g hyp.

PROP. VIII.

In a number AB *be the same parts of a number* CD, *that a part taken away* AE, *is of a part taken away* CF; *the residue also* EB *shall be of the residue* FD *the same parts, that the whole* AB *is of the whole* CD.

```
  6     2    4    2   2
A.....H..G....E..L..B 16
        18         6
C.................F......D 24
```

Divide AB into AG, GB, parts of the number CD; also AE into AH, HE, parts of the number CF; and take $GL=AH=HE$, *a* wherefore $HG=EL$. And because *b* $AG=GB$, *c* therefore $HG=LB$. Now whereas the whole AG is the same part of the whole CD that the part taken away AH is of the part taken away CF, *d* the residue HG or EL shall be the same part also of the residue FD that AG is of CD. In like manner, because GB is the same part of the whole CD, that HE or GL are of CF, *d* therefore the residue LB shall be the same part of the residue FD that GB is of the whole CD. Therefore $EL+LB$ (EB) is the same parts of the residue FD, that the whole AB is of the whole CD. *Which was to be demonstrated.* a 5. ax. b const. c 3. ax. 1. d 7. 7.

Or thus more easily. Let $a+b=x$, and $c+d=y$. Also $y=\frac{2}{3}x$ as well as $c=\frac{2}{3}a$; or, *e* which is the same, $3y=2x$; and $3c=2a$. I say $d=\frac{2}{3}b$. For $3c+3d$ *f* $=3y=2x$ *f* $=2a+2b$. *g* Therefore $3c+3d=2a+2b$ take away from each $3c$ *h* $=2a$, and *k* there remains $3d=2b$, *l* therefore $d=\frac{2}{3}.b$ *Which was to be demonstrated.* e 9. ax. 7. f 1. 2. g 1. ax. 1. h hyp. k 3. ax. 1. l 8. ax. 7.

PROP. IX

A....4
4 4
B....G....C 8
5 D..... 5
5 5
E..... H..... F 10

If a number A be a part of a number BC, and another number D the same part of another number EF; then alternately what part or parts the first A is of the third D, the same part or parts shall the second BC be of the fourth EF.

A is supposed $\sqsupset$ D, therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts
a 1. ax. 7. is put equal in both. But it is clear that BG is *a* the
& 4. 7. same part or parts of EH, that GC is of HF; *b* wherefore
b 5. or 6. 7. BC (BG + GC) is the same part or parts of EF (EH + HF) that BG alone (A) is of EH alone (D.) *Which was to be demonstrated.*

Or thus. Let $a = \frac{b}{3}$, and $c = \frac{d}{3}$; or $3a = b$, and
* 15. 5. $3c = d$, then is $\frac{c}{a}{}^{*} = \left(\frac{3c}{3a} =\right) \frac{d}{b}$.

PROP. X.

A:.G..B 4
C...... 6
5 5
D .....H.....E 10
F 15

If a number AB be parts of a number C, and another number DE the same parts of another number F; then alternately, what parts or part the first AB is of the third DE, the same parts or part shall the second C be of the fourth F.

AB is taken $\sqsupset$ DE, and C $\sqsupset$ F. Let AG, GB, and DH, HE, be parts of the numbers C and F, *viz.* as many in AB as in DE. It is manifest that AG is the
a 9. 7. same part of C, that DH is of F, *a* whence alternately
b 5 & 9. 7. AG is of DH, and likewise GB of HE, and *b* so conjointly AB of DE the same part, or parts, that C is of F. *Which was to be demonstrated.*

Or thus. Let $a = \frac{2b}{3}$, and $c = \frac{2d}{3}$; or $3a = 2b$, and $3c = 2d$. Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

PROP.

PROP. XI.

If a part taken away AE *be to a part taken away* CF, *as the whole* AB *is to the whole* CD, *the residue also* EB *shall be to the residue* FD, *as the whole* AB *is to the whole* CD.

```
    4       3
A . . . . E . . . B  7
      8          6
C . . . . . . . . F . . . . . . D  14
```

First, let AB be $\sqsupset$ CD; *a* then AB is either a part or parts of the number CD; and likewise AE is *b* the same part or parts of CF; *c* therefore the residue EB is the same part or parts of the residue FD that the whole AB is of the whole CD, *b* and so AB. CD : : EB. FD. But if AB be $\sqsubset$ CD, then according to what is already shewn, will CD. AB : : FD. EB, therefore by inversion AB. CD : : EB. FD.

a 4. 7.
b 20. *def.* 7.
c 7. *or* 8. 7.

PROP. XII.

A, 4. C, 2. E, 3.
B, 8. D, 4. F, 6.

If there be numbers, how many soever, proportional (A. B : : C. D : : E. F ;) *then as one of the antecedents* A *is to one of the consequents* B, *so shall all the antecedents* (A+C+E) *be to all the consequents* (B+D+F.)

First, let A, C, E, be $\sqsupset$ B, D, F; then (because of the same proportions) *a* shall A be the same part or parts of B that C is of D; *b* and likewise conjointly A+C shall be the same part or parts of B+D that A alone is of B alone. In the like manner A+C+E is the same part or parts of B+D+F that A is of B. *c* Therefore A+C+E. B+D+F : ; A. B. But if A, C, E, be put greater than B, D, F, the same thing may be shewn by inversion.

a 20. *def.* 7.
b 5, & 6, 7.
c 20. *def.* 7.

PROP. XIII.

If there be four numbers proportional (A. B : : C. D) *then alternately they shall also be proportional*, (A. C : : B. D.)

A, 3. C, 4.
B, 9. D, 12.

First, let A and C be $\sqsupset$ B and D, and A $\sqsupset$ C. By reason of the same proportion *a* shall A be the same part or parts of B, that C is of D. *b* Therefore alter-

a 20. *def.* 7.
b 9, & 10. 7.

nately A is the same part or parts of C that B is of D, and so A. C :: B. D. But if A be ⊏ C, and A and C supposed ⊏ B and D, it will come to the same thing by inverting the proportions.

PROP. XIV.

A, 9. D, 6.
B, 6. E, 4.
C, 3. F, 2.

If there be numbers, how many soever, A, B, C, *and as many more equal to them in multitude, which may be compared two and two in the same proportion* (A. B :: D. E. *and* B. C :: E. F ;) *they shall also by equality, be in the same proportion* (A. C :: D. F.)

5. 7. For because A. B :: D. E, *a* therefore alternately is A. D :: B. E :: *a* C. F ; *a* therefore again, by permutation, A. C :: D. F. *Which was to be demonstrated.*

PROP. XV.

1 . D . .
B . . . 3 . E . . . . . . 6.

If an unite measure any number B, *and another number* D *do equally measure some other number* E; *alternately also shall an unite measure the third number* D, *as often as the second* B *doth the fourth* E.

For seeing 1 is the same part of B, that D is of E ;
a 9. 7. *a* therefore alternately shall 1 be the same part of D, that B is of E. *Which was to be demonstrated.*

PROP. XVI.

B, 4.	A, 3.
A 3.	B, 4.
AB, 12.	BA, 12.

If two numbers A, B, *mutually multiplying themselves, produce any numbers* AB, BA; *the numbers produced* AB, *and* BA, *shall be equal the one to the other.*

a 15. *def.* 7. For Because AB = A x B, *a* therefore shall 1 be as
b 15 7. often in A, as B in AB, *b* and by consequence alternately 1 shall be as often in B as A in AB. But because BA = B x A, *a* therefore shall 1 be as often in B, as A in BA, therefore as often as 1 is in AB, so often is 1 in BA, and
c 4. ax. 7. *c* so AB = BA. *Which was to be demonstrated.*

PROP

PROP. XVII.

If a number A *multiplying two numbers* B, C, *produce other numbers* A B, A C; *the numbers produced of them shall be in the same*	A, 3.	
	B, 2.	C, 4.
	AB, 6.	AC, 12.

proportion that the numbers multiplied are. (AB. AC : : B.C.)

For since AB = A x B, *a* therefore shall 1 be as often in A as B in AB, *a* Likewise because A C = A x C, shall 1 be as often in A, as C in A C, and so also B as often in AB as C in AC; *b* wherefore B. AB : : C. AC, *c* and therefore also alternately B.C : : AB. AC. *Which was to be demonstrated.*

a 15 *def.* 7.
b 20. *def.* 7.
c 13. 7.

PROP. XVIII.

If two numbers AB, *multiplying any number* C, *produce other numbers* A C, B C; *the numbers produced of them shall be in the same*	C, 5.	C, 5.
	A, 3.	B, 9.
	AC, 15.	BC, 45.

proportion that the numbers multiplying are (A. B : : AC. BC.)

For *a* AC = CA, and BC *a* = CB; so the same C multiplying A and B produceth AC and BC, *b* therefore A. B : : AC. BC. *Which was to be demonstrated.*

a 16. 7.
b 17. 7.

Schol.

Hence is deduced the vulgar manner of reducing fractions ($\frac{3}{5}$, $\frac{7}{9}$,) to the same denomination. For multiply 9 both by 3 and 5, and they produce $\frac{27}{45} = \frac{3}{5}$; because by this 3. 5 : : 27. 45. Likewise multiply 5 by 7 and 9, there arises $\frac{35}{45} = \frac{7}{9}$; because 7. 9 : : 35. 45.

PROP. XIX.

If there are four numbers in proportion (A. B : : C. D) *the number produced of the*	A, 4. B, 6. C, 8. D, 12.
	AD, 48. BC 48.

first and fourth (AD) *is equal to the number which is produced of the second and third* (B C.) *And if the number which is produced of the first and fourth* (A D) *be equal to that produced of the second and third* (B C) *those four numbers shall be in proportion* (A. B : : C. D)

1 Hyp.

a 17. 7. 1. *Hyp.* For AC. AD *a* : : C. D *b* : : A. B *c* : : AC. BC,
b *hyp.* *d* therefore AD = BC. *Which was to be dem.*
c 18. 7. 2. *Hyp.* Because *e* AD = BC, therefore AC. AD *f* : :
d 9. 5. AC. BC. But AC. AD *g* : : C. D, and AC. BC *h* : : A.
e *hyp.* B; *k* therefore C. D : : A. B. *Which was to be demon-*
f 17. 5. *strated.*
g 17. 7.
h 18. 7.
k 11. 5.

PROP. XX.

A.	B.	C.
4.	6.	9.
AC, 36.	BB, 36.	
	D, 6.	

If there are three numbers in proportion (A. B : : B. C) *the number contained under the extremes* (AC) *is equal to the square made of the middle* (BB.) *And if the number contained under the extremes be equal to that* (Bq,) *produced of the middle, those three numbers shall be in proportion* $\left(\frac{A}{B} : : \frac{B}{C}\right)$

a 1. *ax.* 7. 1. *Hyp.* For take D = B, *a* therefore A. B : : D (B)
b 19. 7. C; *b* wherefore AC = BD, *a* or BB. *Which was to be demonstrated*
c *hyp.* 2. *Hyp* Because AC *c* = BD, *d* therefore A. B : : D
d 19. 7. (B) C. *Which was to be demonstrated.*

PROP. XXI.

A . . . G . . B 5. E . . . . 10.
C . . H . D 3. F . . . . . . 6.

Numbers AB, CD, *being the least of all that have the same proportion with them* (E, F,) *do equally measure the numbers* E, F, *having the same proportion with them; the greater* AB *the greater* E, *and the lesser* CD *the lesser* F.

For AB. CD *a* : : E. F, *b* therefore alternately AB. E : :
a *hyp.* CD. F, *c* therefore AB is the same part or parts of E that
b 13. 7. CD is of F; but parts it cannot be, for if so, then let
c 20. *def.* 7. AG, GB, be parts of the number E; and CH, HD, parts of the number F, *c* therefore AG. E : : CH. F, and by inversion AG. CH *d* : : E. F *e* : : AB. CD; there-
d 13. 7. fore AB, CD, are not the least in their proportion;
e *hyp.* which is contrary to the hypothesis. Therefore, &c.

PROP.

PROP. XXII.

If there are three numbers A, B, C; *and other numbers equal to them in multitude,* D, E, F; *which may be compared two and two in the same proportion: and if also the proportion of them be perturbed* (A. B :: E. F. *and* B. C :: D. E) *then by equality they shall be in the same proportion* (A. C :: D. F.)

A, 4.	D, 12.
B, 3.	E. 8.
C, 2.	F, 6.

For because A. B *a* :: E. F, therefore shall AF = BE; and because B. C :: *a* D. E, *b* therefore BE = CD, *c* and consequently AF = CD. *d* Therefore A. C :: D. F. *Which was to be demonstrated.* a *hyp.* b 19. 7. c 1. *ax.* 1. d 19. 7.

PROP. XXIII.

Numbers prime the one to the other, A, B, *are the least of all numbers that have the same proportion with them.*

A, 9. B, 4.
C ---- D ---
E ---

If it be possible, let C and D be less than A and B, and in the same proportion; *a* therefore C measures A equally as D measures B, suppose by the same number F; and so C shall be *b* as often in A as 1 is in E; *c* likewise alternately, E as often in A as 1 in C. By the like reasoning as many times as 1 is in D, so many times shall E be in B. Therefore E measures both A and B; which consequently are not prime the one to other, *contrary to the hypothesis.* a 21. 7. b 23. *def.* 7. c 15. 7.

PROP. XXIV.

Numbers A, B, *being the least of all that have the same proportion with them, are prime the one to the others.*

A, 9. B, 4.
C ---
D --- E --

If it possible, let A and B have a common measure C; and let the same measure A by D, and B by E; *a* therefore CD = A, *b* and CE = B. *b* Wherefore A. B :: D. E. But D and E are lesser than A and B, as being but parts of them. Therefore A and B are not the least in their proportion, *against the Hypothesis.* a 9. *ax.* b 17.

PROP.

PROP. XXV.

A, 9. B, 4.
C, 3. D- -

If two Numbers A, B, are prime the one to the other, the number C measuring one of them A, shall be prime to the other number B.

a 11. ax. 7. For if you affirm any other D to measure the numbers B and C, *a* then D measuring C does also measure A; and consequently A and B are not prime the one to the other: *Which is against the Hypothesis.*

PROP. XXVI.

A, 5. C, 8.
B, 3.
AB, 15. E- - - -
F- - - -

If two numbers A, B, are prime to any number C, the number also produced of them AB, shall be prime to the same C.

If it be possible, let the number E be a common measure to AB, and C; and let $\frac{AB}{E}$ be $=$ F; *a* thence AB $=$ EF; *b* wherefore also E. A : : B. F. But because A is prime to C, which E measures, *c* therefore E and A are prime to one another, *d* and so least in their own proportion, *e* and consequently they must measure B and F; namely F shall measure B, and A shall measure F. Therefore seeing E measures both B and C, they shall not be prime to one another: *Contrary to the Hypothesis.*

a 9. ax. 7.
b 19. 7.
c 25. 7.
d 23. 7.
e 21. 7.

PROP. XXVII.

A, 4. B, 5.
Aq, 16.
D, 4.

If two numbers A, B, are prime to one another, that also which is produced of one of them (Aq) shall be prime to the other B.

a 1 ax. 7. Take D $=$ A; therefore both D, and A are prime to B; *b* therefore AD or Aq is prime to B. *Which was to be demonstrated.*

b 26. 7.

PROP.

PROP. XXVIII.

If two numbers A, B, *are prime to two numbers* C, D, *each to either of both, the numbers also produced by multiplying them* AB, CD, *shall be prime to one another.*

A, 5. C, 4
B, 3. D, 2.
AB, 15. CD, 8.

For because A and B are prime to C, *a* therefore shall AB also be prime to the same And for the same reason shall AB be prime to D. *b* Therefore AB is prime to CD. *Which was to be demonstrated.* a 26. 7. b 26. 7.

PROP. XXIX.

If two Numbers A, B, *are prime to one another, and each multiplying itself produces another number* (Aq, *and* Bq;) *then the numbers produced of them* (Aq, Bq,) *shall be prime to one another. And if the numbers given at first* A, B, *multiplying the said produced numbers* (Aq, Bq,) *produce others* (Ac, Bc,) *those numbers also shall be prime to one another: And so on.*

A, 3.	B, 2.
Aq, 9.	Bq, 4.
Ac, 27.	Bc, 8.

For because A is prime to B, *a* therefore Aq shall be prime to B, and Aq being prime to B, *a* therefore Aq shall be also prime to Bq. Again, because A is as well prime to B and Bq, as Aq is to the said B and Bq, *b* therefore shall A x Aq, that is, Ac, be prime to B x Bq, that is, to Bc: And so forth of the rest. a 27. 7. b 28. 7.

PROP. XXX.

If two numbers AB, BC, *be prime the one to the other; then both added together* (AC) *shall be prime to either of them* AB, BC. *And if both added together* AC *be prime to any one of them* AB, *the numbers also given in the beginning* AB, BC, *shall be prime to one another.*

8 5
A .... B ..... C 13. D —

1. *Hyp* For if you would have AC, AB to be composed, let D be the common measure: *a* this shall measure the residue BC: And therefore AB, BC, are not prime to one another; *which is against the Hypothesis.* a 12. ax. 7.

2 *Hyp* AC, AB, being taken prime to one another, let D be the common measure of AB, BC. *b* But seeing that measures the whole AC, therefore AC, AB, are not prime to one another; *contrary to the Hypothesis,* b 10. ax. 7.

Coroll.

Coroll.

Hence, A number, which being compounded of two, is prime to one of them, is also prime to the other.

PROP. XXXI.

A, 5. B, 8. *Every prime number* A *is prime to every number* B, *which it measureth not.*

For if any common measure doth measure both, A, B, *a* then A will not be a prime number; *contrary to the Hypothesis.*

a 11 *def* 7.

PROP. XXXII.

A, 4.	D, 3.
B, 6.	E, 8.
AB, 24.	

If two numbers A, B, *multiplying one another produce another* AB, *and some prime Number* D, *measure the number produced of them* AB; *then shall it also measure one of those numbers,* A, *or* B, *which were given at the beginning.*

Suppose the number D not to measure the number A, and let $\frac{AB}{D}$ be $=$ E, *a* then AB $=$ DE; *b* whence D. A : : B. E. *c* But D is prime to A; *d* therefore D and A are the least in their proportion; *e* and consequently D measures B as often as A measures E. *The proposition therefore is evident.*

a 9. *ax.* 7.
b 19. 7.
c *hyp. and* 31. 7.
d 23. 7.
e 21. 7.

PROP. XXXIII.

A, 12. *Every composed number* A, *is measured by*
B, 2. *some prime number* B.

Let one or more numbers *a* measure A, of which let the least be B; that shall be a prime number: For if it be said to be composed, then some *a* lesser number shall measure it, *b* which shall also consequently measure A. Wherefore B is not the least of those which measure A, *contrary to the Hyp.*

a 13 *def.* 7.
b 11 *ax.* 7.

PROP.

PROP. XXXIV.

A, 9. *Every number* A, *is either a prime, or measured by some prime number.*

For A is necessarily either a prime or a composed number. If it be a prime, 'tis that we affirm. If composed, *a* then some prime number measureth it. *Which was to be demonstrated* a 33. 7.

PROP. XXXV.

A, 6. B, 4. C, 8.
D, 2. H.. I.. K....
E, 3. F, 2. G, 4. L...

How many numbers soever A, B, C, *being given to find the least numbers* E, F, G, *that have the same proportion with them.*

If A, B, C, be prime to one another, *a* they shall be the least in their proportion. If they be composed, *b* let their greatest common measure be D, which let measure them by E, F, G. These are then the least in the proportion A, B, C. a 23. 7. b 3. 7.

For D x E, F, G, *c* produceth A,B,C, *d* therefore these and those are in the same proportion. But allow other numbers H, I, K, to be the least in the same proportion; *e* which shall therefore equally measure A, B, C, namely by the number L, *f* therefore L x H, I, K, shall produce A, B, C, *g* and consequently ED = A = HL; *h* from whence E. H : : L. D But E *k* ⊏ H; *l* therefore L ⊏ D, and so D is not the greatest common measure of A, B, C. *Which is against the Hypothesis.* c 9. *ax*. 7. d 17. 7. e 21. 7. f 9 *ax*. 7. g 1. *ax*. 1. h 19. 7. k *suppos.* l 20. *def*. 7.

Coroll.

Hence, The greatest common measure of how many numbers soever, does measure them by the numbers which are least of all that have the same proportion with them. Whereby appears the vulgar method of reducing fractions to the least terms.

PROP.

PROP. XXXVI.

Two numbers being given, A, B, *to find out the least number which they measure.*

A, 5. B, 4.
AB, 20.
D - - - - - -
E - - - F - - - -

1. *Case.* If A and B be prime the one to the other, AB is the number required. For it is manifest that A and B measure AB. If it be possible, let A and B measure some other number D
a 9. ax. 7. & 1. ax. 1. —⊐ AB, suppose by E, and F, *a* therefore AE = D = BF, *b* and so A. B : : F. E. But because A and B *c* are
b 19. 7. prime the one to other, *d* and so least in their proportion,
c hyp. A shall *e* equally measure F as B does E. But B. E *f* : :
d 23. 7. AB. AE (D.) *g* Therefore AB shall also measure D,
e 21. 7. which is less than it self. *Which is absurd.*
f 17. 7.
g 20. def. 7.

A, 6. B, 4. F - - - -
C, 3. D, 2. G - - . H - - -
AD, 12.

2. *Case.* But if A and B be composed one to another,
h 35. 7. *h* let there be found C and D the least
k 19. 7. in the same proportion. *k* Therefore AD = BC; and AD or BC shall be the number sought.
l 7. ax. 7. For it is *l* plain that B and A do measure AD or BC. Conceive A and B to measure F —⊐ AD, namely A by
m 9. ax. 7. G, and B by H, *m* therefore AG = F = BH, *n* whence
n 19. 7. A. B : : H. G *o* : : C. D, *p* and consequently C equally
o constr. measures H as D does G. But D. G *q* : : AD. AG (F,)
p 21. 7. therefore AD *r* measures F, the greater the less. *Which*
q 17. 7. *is absurd.*
r 20 def. 7.

Coroll.

Hence, If two numbers multiply the least that are in the same proportion, the greater the less, and the less the greater, the least number which they measure shall be produced.

PROP. XXXVII.

A, 2 B, 3.
E 6
C - - - - F - - - D

If two numbers A, B, *measure any number* CD, *the least number which they measure* E *shall also measure the same* CD.

If you deny it, take E from CD as often as you can,
a hyp. and leave FD —⊐ E, therefore seeing A and B *a* measure
b constr E, *b* and E measures CF, *c* likewise A and B will measure
c 11. ax. 7. CF. But *a* they measure the whole CD; *d* therefore also
d 12 ax. 7. they measure the residue FD; and consequently E is not the least which A and B measure: *Contrary to the Hypothesis.*

PROP.

PROP. XXXVIII.

Three numbers being given, A, B, C, to find out the least which they measure.

A, 3. B, 4. C, 6.
D, 12.

a Find D the least that two of them A and B do measure; which if the third C do also measure, it is manifest that D is the number sought. But if C doth not measure D, let E be the least that C and D do measure, E shall be the number required. a 36. 7.

For it appears by the 11. *ax.* 7. that A, B, C, measure E; and it is easily shewn that they measure no other F less than E.

A, 2. B, 3. C, 4.
D, 6. E, 12.
F - - -

For if you affirm they do, *b* then D measures F, *b* and consequently E measures the same F, the greater the less. *Which is absurd.* b 37. 7.

Coroll.

Hence it appears, that if three numbers measure any number, the least also, which they measure, shall measure the same.

PROP. XXXIX.

If any number B measures a number A, the number measured A, shall have a part C denominated of the number measuring B.

A, 12.
B, 4. C, 3.

For because $\frac{A}{B}$ *a* $=$ C, *b* shall A $=$ BC, *c* therefore $\frac{A}{C}$ $=$ B. *Which was to be demonstrated.*

a hyp.
b 9. ax. 7.
c 7. ax. 7.

PROP. XL.

If a number A have any part whatsoever B, the number C, from which the part B is denominated, shall measure the same.

A, 15.
B, 3. C, 5.

For since BC *a* $=$ A, *b* shall $\frac{A}{C}$ $=$ B. *Which was to be demonstrated.*

a hyp. & 9 ax. 7.
b 7. ax. 7.

PROP. XLI.

To find out a number G, the least that can have given parts, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

$\frac{1}{2}$ G, 12.
$\frac{1}{3}$ H - - -
$\frac{1}{4}$

a Let G be found the least which the denominators 2, 3, 4, measure; *b* it is evident that G has the parts $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. If it be possible let H $\sqsubset$ G have the same parts; *c* therefore 2, 3, 4, measure H; and so G is not the least which 2, 3, 4, measure: *against the constr.*

a 38. 7.
b 39. 7.
c 40. 7.

The End of the seventh Book.

The Eighth Book OF *EUCLIDE's* ELEMENTS.

PROP. I.

A, 8. B, 12. C, 18. D, 27.
E - F -- G --- H ----

If there be divers numbers how many soever in continual proportion, A, B, C, D, *and their extremes* A, D, *prime to one another*; *then those numbers* A, B, C, D, *are the least of all numbers that have the same proportion with them.*

For, if it be possible, let there be as many others E, F, G, H, less than A, B, C, D, and in the same proportion with them. *a* Therefore from equality A. D : : E. H, and so A and D which are prime numbers, *b* and consequently the least in their proportion, *c* equally measure E and H, which are less than themselves. *Which is absurd.*

a 14. 7.
b 23. 7.
c 21. 7.

PROP. II.

1
A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.
Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the least numbers continually proportional, as many as shall be required, in the proportion given of A *to* B.

Let A and B be the least in the proportion given; Then Aq, AB, Bq, shall be the three least in the same continual proportion that A is to B.

For AA. AB *a* : : A. B *a* : : AB. BB. Likewise because A and B are prime one to another, *c* shall Aq, Bq, be also prime to one another, *d* and so Aq, AB, Bq, are the least in the proportion of A to B.

a 17. 7.
b 24. 7.
c 29. 7.
d 1. 8.

More-

Moreover, I say Ac, AqB, ABq, Bc, are the four least in the proportion of A to B. For AqA. AqB *e* : : A. B. *e* : : ABA (AqB.) ABB. *e* and A. B : : ABq. BBq (Bc.) Therefore since Ac, and Bc, are *f* prime to one another, likewise *g* shall Ac, AqB, ABq, Bc be the four least ∺ in the proportion of A to B. In the same manner may you find out as many proportional numbers as you please. *Which was to be done.* e 17. 7. f 29. 7. g 1. 8.

Coroll.

1. Hence, If three numbers, being the least, are proportional, their extremes shall be squares; if four, cubes.

2. The extremes of any number of proportionals found by this proposition, if such proportionals are the least of all in a given ratio, are prime to one another.

3. Two numbers, being the least in a given ratio, do measure all the mean numbers of proportionals, be they ever so many, provided they are the least in the same proportion; because they arise from the multiplication of them into certain other numbers.

4. Hence it also appears by the construction, that the series of numbers 1, A, Aq, Ac; 1, B, Bq, Bc; Ac, AqB, ABq, Bc, consist of an equal multitude of numbers; and consequently, the extreme numbers of how many soever the least continually proportionals are the last of as many other continually proportionals from unity; thus the extremes Ac, Bc, of the continually proportionals Ac, AqB, ABq, Bc, are the last of as many proportionals from unity 1, A, Aq, Ac, and 1, B, Bq, Bc.

5. 1, A, Aq, Ac; and B, BA, BAq; and Bq, ABq are ∺ in the ratio of 1 to A. Also B, Bq, Bc; and A, AB, ABq; and Aq, AqB are ∺ in the ratio of 1 to B.

PROP. III.

If there be numbers continually proportional, A, 8. B, 12. C, 18. D, 27. *how many soever,* A, B, C, D, *being also the least of all that have the same proportion with them; their extremes* A, D, *are prime to one another.*

For if there be *a* found as many numbers the least in the proportion of A to B, they shall be no other than A, B, C, D; therefore, by the second *Coroll.* of the precedent *prop.* the extremes A and D are prime to one another. *Which was to be demonstrated.* a 2. 8.

PROP. IV.

A, 6. B, 5. C, 4. D, 3.
H, 4. F, 24. E, 20. G, 15.
I - - K - - L - - -

Proportions how many soever being given in the least numbers (A, to B, and C to D) to find out the least numbers continually proportional in the proportions given.

a 36. 7. *a* Find out E the least number which B and C do measure; and let B measure E *b* as often as A does another
b 3 *post.* 7. F, *viz.* by the same number H. *b* Also let C measure the said E as often as D measures another G, then F, E, G, shall
c 9. *ax.* 7. be the least in the proportions given. For AH *c* = F,
d 18. 7. and BH *c* = E; *d* therefore A. B : : AH. BH *e* : : F. E. In
e 7. 5. like manner C. D : : E. G; therefore F, E, G, are continually proportional in the proportions given. And they are moreover the least in the said proportions; for con-
f 21. 7. ceive other numbers I, K, L, to be the least; *f* then A and B must equally measure I and K, *f* and C and D likewise K and L; and so B and C measure the same
g 37. 7. K. *g* Wherefore also E measures the same number K, which is less than it self. *Which is absurd.*

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15 K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the least continually in the proportions of A to B, and C
h 3 *post.* 7. to D. Then if E measures I, *h* take another number K, which may be equally measured by F; and those four numbers H, G, I, K, shall be continually the least in the given proportions; which we need go no other way to prove than we did in the first part.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 20. I, 15.
M, 48. L, 40. K, 30. N, 105.

If E doth not measure I, let K be the least which E and I do measure; and as often as I measures K, let G as often measure L, and H also M, so likewise let F measure N as often E measures K. The four numbers M, L, K, N, shall be least continually in the given proprortions; which we may demonstrate as before.

PROP.

PROP. V.

Plain numbers CD, EF, *are in that proportion to one another which is composed of their sides.*

C, 4. E, 3.
D, 6. F, 16. ED, 18.
CD, 24. EF, 48.

For because CD. DE *a* : : C. E; *a* and ED. EF : : D. F. a 17. 7.
and $\frac{CD}{EF}$ *b* $= \frac{CD}{ED} \times \frac{ED}{EF}$ *c* then shall be the proportion b 20 *def.* 5. c 11. 5.
$\frac{CD}{EF} = \frac{C}{E} \times \frac{D}{F}$. *Which was to be demon.*

PROP. VI.

If there be numbers continually proportional how many soever, A, B, C, D, E, *and the first* A *does not measure the second* B, *neither shall any of the other measure any one of the rest.*

A, 16. B, 24. C, 36. D, 54. E. 81.
F, 4 G, 6. H, 9.

Since A does not measure B, *a* neither shall any one measure that which next follows; Because A. B : : B. C : : C. D, &c. *b* Take three numbers, F, G, H, the least in the proportion of A to B, therefore since A does not measure B, *a* neither shall F measure G, *c* therefore F is not unity. But F and H are prime one to another; therefore *d* since by equality A. C : : F. H, and F does not measure H, *a* neither shall A measure C; and consequently neither shall B measure D, nor C measure E, &c. because A. C *e* : : B. D *e* : : C. E, &c. In like manner four or five numbers being taken the least in the proportion of A to B, it may be shewn that A does not measure D and E; nor does B measure E and F, &c. Wherefore none of them shall measure any other. *Which was to be demonstrated.*

a 20. *def.* 7. b 35 7. c 5. 7. d 3. 3. e 14. 7.

PROP. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportional how many soever A, B, C, D, E, *and the first* A *measures the last* E, *it shall also measure the second* B.

If you deny that A measures B, *a* then neither shall it measure E; *Which is contrary to the Hyp.* a 6. 7.

PROP. VIII.

A, 24. C, 36. D, 54. B, 81.
G, 8. H, 12. I, 18. K, 27
E, 32. L, 48. M, 72. F, 108.

If between two numbers A, B, *there fall mean numbers in continual proportion* C, D; *as many mean numbers in continual proportion as fall between them, so many mean numbers also* L, M, *in continual proportion, shall fall between two other numbers* E, F, *which have the same proportion with them* (L. M.)

a 35. 7. *a* Take G, H, I, K, the least ∺ in the proportion of
b 14. 7. A to C; *b* by equality it shall be G. K : : A. B *c* : : E. F.
c *hyp.* But G, and K *d* are prime one to another. *e* Wherefore
d 3. 8. G measures E as often as K does F. Let H measure L,
e 21. 7. and I likewise M by the same number; *f* therefore E,
f *constr.* L, M, F, are in such proportion as G, H, I, K, that is, as A, B, C, D. *Which was to be demonstrated.*

PROP. IX.

1.
E, 2. F, 3.
G, 4. H, 6. I, 9.
A, 8. C, 12. D, 18. B, 27.

If two numbers A, B *are prime to one another, and mean numbers in continual proportion* C, D, *fall between them; as many mean numbers in continual proportion as fall between them, so many means also in continual proportion* (E, G; and F, I) *shall fall between either of them and unity.*

It is evident, that 1, E, G, A, and 1, F, I, B, are ∺, and as many as A, C, D, B, namely by the 4th Coroll. 2. 8. *Which was to be demonstrated.*

PROP. X.

A, 8. I, 12. K, 18. B. 27.
E, 4. DF, 6. G, 9.
D, 2. F, 3.
1.

If between two numbers A, B, *and an unit, numbers continually proportional* (E, D, and F, G, *do fall, how many mean numbers in continual proportion fall between either of them and unity, so many means also shall fall in continual proportion between them,* I, K.

For E, DF, G, and A, DqF (I) DG (K) B are ∺ by 2. 8, therefore, &c.

PROP.

PROP. XI.

A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.

Between two square numbers Aq, Bq, *there is one mean proportional number* AB: *and* Aq *to* Bq, *is in duplicate proportion of the side* A *to the side* B.

a It is manifest that Aq, AB, Bq, are ∺; *b* and consequently also $\frac{Aq}{Bq} = \frac{A}{B}$ twice. *Which was to be demonstrated.* a 17. 7. b 10. *def.* 5.

PROP. XII.

Ac, 27. AqB, 36. ABq, 48. Bc, 64.
A, 3. B, 4.
Aq, 9. AB, 12. Bq, 16.

Between two cube numbers, Ac, Bc, *there are two mean proportional numbers* AqB, ABq: *and the cube* Ac *is to the cube* Bc *in triplicate ratio of the side* A *to the side* B.

a For Ac, AqB, ABq, Bc, are ∺ in the proportion of A to B; *b* and therefore $\frac{Ac}{Bc} = \frac{A}{B}$ thrice. *Which was to be demonstrated.* a 2. 8. b 10. *def.* 5.

PROP. XIII.

A, 2. B, 4. C, 8.
Aq, 4. AB, 8. Bq, 16. BC, 32. Cq, 64.
Ac, 8. AqB, 16. ABq, 32. Bc, 64. BqC, 128. BCq, 256. Cc, 512.

If there be numbers in continual proportion how many soever A, B, C; *and every of them multiplying it self produces certain numbers; the numbers produced of them* Aq, Bq, Cq, *shall be proportional: And if the numbers first given* A, B, C, *multiplying their products* Aq, Bq, Cq, *produce other numbers* Ac, Bc, Cc, *they also shall be proportional; and this shall ever happen to the extremes.*

For Aq, AB, Bq, BC, Cq *a* are ∺; *b* therefore by equality Aq. Bq :: Bq. Cq. *Which was to be demonstrated.* a 2. 8. b 14. 7.

a Also Ac, AqB, ABq, Bc, BqC, BCq, Cc, are ∺; *b* therefore again by equality Ac. Bc :: Bc. Cc. *Which was to be demonstrated.*

PROP. XIV.

Aq, 4. AB, 12. Bq, 36.
A, 2. B, 6.

If a square number Aq *measure a square number* Bq, *the side also of the one* (A) *shall measure the side of the other* (B): *and if the side of one square* A *measure the side of another* B, *the square* Aq *shall likewise measure the square* Bq.

a 12 & 11.8. 1. *Hyp.* For Aq. AB *a* : : AB. Bq, therefore seeing by
b 7.8. the hypothesis Aq measures Bq, *b* it shall measure also
c 20.*def.* 7. AB. But Aq: AB : : A. B, *c* therefore also A measures B. *Which was to be demonstrated*

2. *Hyp.* A measures B, *c* therefore Aq shall as well
d 11. *ax.* 7. measure AB, *c* as AB measures Bq ; *d* consequently Aq measures Bq. *Which was to be demonstrated.*

PROP. XV.

A, 2. B, 6.
Ac, 8. AqB, 24. ABq, 72. Bc, 216.

If a cube number Ac *measures a cube number* Bc, *then the side of the one* (A) *shall measure the side of the other* (B.) *And if the side* A *of one cube* Ac *measure the side* B *of the other* BC, *also the cube* Ac *shall measure the cube* Bc.

a 2.&12.8. 1. *Hyp.* For Ac, AqB, ABq, Bc *a* are ∺, therefore
b *hyp.* Ac, *b* measuring the extreme Bc, shall also *c* measure the
c 7.7. second AqB. But Ac. AqB : : A. B, *d* therefore A shall also measure B. *Which was to be dem.*

d 20.*def.* 7. 2. *Hyp.* A measures B ; *d* therefore Ac measures
e 11. *ax.* 7. AqB, which also measures ABq, and this Bc ; *e* therefore Ac shall measure Bc. *Which was to be demonstrated.*

PROP. XVI.

A, 4. B, 9.
Aq, 16. Bq, 81.

If a square number Aq *do not measure a square number* Bq, *neither shall the side of the one* A *measure the side of the other* B : *And if* A *the side of the one square* Aq *do not measure* B *the side of the other* Bq, *neither shall the square* Aq *measure the square* Bq.

1. *Hyp.* For if you affirm that A measures B, then
a 14.8. Aq also shall measure Bq *Against the Hypothesis.*

2 *Hyp.* If you maintain Aq to measure Bq ; *a* then likewise A shall measure B, *Contrary to the Hypothesis.*

PROP.

PROP. XVII.

A, 2.	B, 3.
Ac, 8.	Bc, 27.

If a cube number Ac *does not measure a cube number* Bc, *neither shall the side of one* A *measure the side of the other* B : *And if* A *the side of one cube* Ac *does not measure* B *the side of the other* Bc, *neither shall the cube* Ac *measure the cube* Bc.

1. *Hyp.* Let A measure B; *a* then Ac shall measure Bc. *Against the Hypothesis.* a 15. 8.

2. *Hyp.* Let A measure Bc; then A shall measure B; *which is also against the Hypothesis.*

PROP. XVIII.

C, 6. D, 2.
CD, 12.
E, 9. F, 3. DE, 18.
EF, 27.

Between two like plane numbers CD *and* EF *there is one mean proportional number* DE : *And the plane* CD *is to the plane* EF *in duplicate proportion of that which the side* C *hath to the homologous side* E.

For * by the Hypothesis C. D : : E. F; therefore by permutation, C. E : : D. F. But C. E *a* : : CD. DE; *a* and D. F : : DE. EF; *b* therefore CD. DE : : DE. EF. *c* Wherefore the proportion of CD to EF is duplicate to that of CD to DE, that is, to the proportion of C to E, or D to F. * 21 *def.* 7. a 17. 7. b 11. 5. c 10. *def.* 5.

Coroll.

Hence it is apparent, That between two like plane numbers there falls one mean proportional in the proportion of the homologous sides.

PROP. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.
CD, 6. DF, 12. FG, 24.
C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

Between two like solid numbers CDE, FGH, *there are two mean proportional numbers* DFE, FGE. *And the solid* CDE *is to the solid* FGH, *in triplicate proportion of that which the homologous side* C *has to the homologous side* F.

Where-

*21 *def.* 7. Whereas by the * hyp. C. D : : F. G, and D. E : : G
a 13. 7. H, therefore *a* by permutation, shall C. F : : D. G : : *a*
b 17. 7. E. H. But CD. DF *b* : : C. F, and DF. FG *b* : : D. G;
c 11. 5. *c* wherefore CD. DF : : DF. FG : : E. H; *d* therefore
d 17. 7. CDE. DFE : : DFE. FGE : : E. H : : FGE. FGH. There-
e 10. *def.* 5. fore between CDE, FGH, fall two mean proportionals DFE, FGE. *e* And so it is plain that the proportion of CDE to FGH is triplicate to that of CDE to DFE, or C to F. *Which was to be demonstrated.*

Coroll.

Hereby it is manifest, that between two like solid numbers there fall two mean proportionals in the proportion of the homologous sides.

PROP. XX.

A, 12. C, 18. B, 27.
D, 2. E, 3. F, 6. G, 9.

If between two numbers A, B, *there falls one mean proportional number* C; *those numbers* A, B, *are like plane numbers.*

a 35. 7. *a* Take D and E the least in the proportion of A to C, or C to B, then D measures A equally as E does C; suppose by the same number F; *b* also D equally measures
b 21. 7. C, as E does B, suppose by the same number G. *c* There-
c 9. *ax.* 7. fore DF = A, and EG = B, *d* and consequently A and B are
d 16 *def.* 7. plane numbers. But because EF *c* = C *c* = DG, *e* shall
e 19. 7. D. E : : F. G, and alternately D. F : : E. G. *f* Therefore
f 21. *def.* 7. the plane numbers A and B are also like. *Which was to be demonstrated.*

PROP. XXI.

A, 16. C, 24. D, 36. B, 54.
E, 4. F, 6. G, 9.
H, 2. P, 2. M, 4. K, 3. L, 3. N, 6.

If between two numbers A, B, *there fall two mean proportional numbers* C, D; *those numbers* A, B, *are like solid numbers.*

a 2. 8. *a* Take E, F, G, the least ∺ in the proportion of A
b 10. 8. to C, *b* then D and G are like plane numbers: let the
c 21 *def.* 7. sides of this be H and P, and of that K and L, *c* there-
d *cor.* 18. 8. fore H. K : : P. L : : *d* E. F. But E, F, G, do *e* equally
e 21. 7. measure A, C, D, suppose by the same number M, and likewise the said numbers E, F, G, do equally measure the numbers C, D, B, suppose by the same number N.
f 9. *ax.* 7. *f* Therefore A = EM = HPM, *f* and B = GN = KLN;

g and

g and so A and B are solid numbers. But because C f = FM, and D f = FN, therefore shall M. N h :: FM. FN k :: C. D l :: E. F :: H. K :: P. L; m therefore A and B are like solid numbers. *Which was to be demonstrated.* g 17 *def.* 7. h 17. 7. k 7. 5. l *constr.* m 21. *def.* 7.

Lemma.

If proportional numbers A, B, C, D, *measure proportional numbers* AE, BF, CG, DH, *by the numbers* E, F, G, H, *these numbers* (E, F, G, H) *shall be proportional.*

AE, BF, CG, DH,
A, B, C, D,
E, F, G, H,

For because AEDH a = BFCG, a and AD = BC, b shall $\frac{AEDH}{AD} = \frac{BFCG}{BC}$ c that is, EH = FG. a Therefore E. F :: G. H. *Which was to be demon.* a 19. 7. b 1 *ax.* 7. c 9 *ax.* 7.

Coroll.

Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. d For 1. B :: B. Bq, d and 1. A :: A. Aq, e therefore 1. $\frac{B}{A}$ :: $\frac{B}{A}$. $\frac{Bq}{Aq}$, d therefore $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. In like manner $\frac{B}{Ac} \times \frac{Bq}{Ac} = \frac{Bc}{Acc}$ and so of the rest. d 15. *def.* 7. e *lem. prec.*

PROP. XXII.

If three numbers Aq, B, C, *are continually proportional, and the first* Aq *a square, the third* C *shall also be a square.*

Aq, B, C,
4, 8, 16.

For because Aq C a = Bq, b thence is C = $\frac{Bq}{Aq}$ c = Q. $\frac{B}{A}$. But it is plain that $\frac{B}{A}$ is a number, d because $\frac{Bq}{Aq}$, or C is a number. Therefore if three, &c. a 20 7. b 7. *ax.* 7. c *cor. of the lem. prec.* d *hyp. and* 14 8.

PROP. XXIII.

If four numbers A, B, C, D, *are continually proportional, and the first of them* Ac *a cube, the fourth also* D, *shall be a cube.*

Ac, B, C, D
8, 12, 18, 27.

For

a 19. 7. For because $AcD \overset{a}{=} BC$, *b* therefore $D = \frac{BC}{Ac}$
b 7. ax 7.
c cor. of the prec. lem. $\overset{c}{=} \frac{B}{Ac} \times C$; that is, (because $AcC \overset{d}{=} Bq$, and
d 20. 7. *b* thence $C = \frac{Bq}{Ac}$) $D = \frac{B}{Ac} \times \frac{Bq}{Ac} \overset{e}{=} \frac{Bc}{Acc} \overset{c}{=} C : \frac{B}{Ac}$.

e 15. 8. But it is evident *e* that $\frac{B}{Ac}$ is a number, because $\frac{Bc}{Acc}$ or D is supposed a number. Therefore if four numbers, &c.

PROP. XXIV.

A, 16. 24. B, 36.
C, 4. 6. D, 9.

If two numbers A, B, *be in the same proportion one to another, that a square number* C *is to a square number* D, *and the first* A *be a square number, the second also* B *shall be a square number.*

* 8. 8. Between C and D the square numbers, * and so between A and B having the same proportion, *a* falls one
a 11. 8. mean proportional. Therefore *b* since A is a square
b hyp. number, *c* B also shall be a square number. *Which was*
c 22. 8. *to be demonstrated.*

Coroll

1. Hence, if there be two like numbers AB, CD, (A. B : : C. D) and the first AB be a square, the second also CD shall be a square.

* 11. and 18. 8. * For AB. CD : : Aq. Cq.

2. From hence it appears, That the proportion of any square number to any other not square, cannot possibly be declared into two square numbers. Whence it cannot be Q. Q : : 1. 2, nor 1. 5 : : Q. Q, &c.

PROP. XXV.

C, 64. 96. 144. D, 216.
A, 8. 12. 18. B, 27.

If two numbers A, B, *are in the same proportion one to another, that a cube number* C *is to a cube number* D, *the first of them* A *being a cube number; the second* B *shall likewise be a cube number.*

a 12. 8. *a* Between the cube numbers C and D, *b* and so between
b 8. 8. A and B having the same proportion, fall two mean pro-
c hyp. portionals; therefore *c* because A is a cube, *d* shall B be
d 23. 8. a cube also. *Which was to be demonstrated.*

Coroll.

Coroll.

1. Hence, If there be two numbers ABC, DEF, (A. B :: D. E, and B. C :: E. F;) and the first ABC be a cube, the second DEF shall be a cube also.

* For ABC. DEF :: Ac. Dc. * 12, *and* 19. 8.

2. Hence it is evident, That the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

PROP. XXVI.

Like plane numbers A, B, *are in the same proportion one to another, that a square number is in to a square number.*

A, 20.	C, 30.	B, 45.
D, 4.	E, 6.	F, 9.

Between A and B *a* falls one mean proportional number C; *b* take three numbers D, E, F, the least ∺ in the proportion of A to C, the extremes D, F, *c* shall be square numbers. But by equality A. B *d* :: D. F, therefore A. B :: Q. Q. *Which was to be demonstrated.* a 18. 8. b 2. 8. c *cor.* 2. 8. d 14. 7.

PROP. XXVII.

Like solid numbers A, B, *are in the same proportion one to another, that a cube number is to a cube number.*

A, 16.	C, 24.	D, 36.	B, 54.
E, 8.	F, 12.	G, 18.	H, 27.

a Between A and B fall two mean proportional numbers, namely, C and D: *b* take four numbers E, F, G, H, the least ∺ in the proportion of A to C, *b* the extremes E, H, are cube numbers. But A. B *c* :: E. H :: C. C. *Which was to be demonstrated.* a 19. 8. b 2. 8. c 14. 7.

Schol.

1. From hence is inferred, that no numbers in proportion superparticular, or superbipartient, or double, or any other manifold proportion not denominated from a square number, are like plane numbers. See *Clavius.*

2. Likewise, that neither any two prime numbers, nor any two numbers prime one to another, not being squares, can be like plane numbers.

The End of the eighth Book.

THE

The NINTH BOOK OF EUCLIDE's ELEMENTS.

PROPOSITION I.

A, 6. B, 54.
Aq, 36. 108. AB, 324.

If two like plane numbers A, B, *multiplying one another, produce a number* AB, *the number produced* AB *shall be a square number.*

a 17. 7. For A. B *a* : : Aq. AB; wherefore since one mean pro-
b 18. 8. portional *b* falls between A and B, *c* likewise one mean
c 8. 8. proportional number shall fall between Aq and AB:
d 22. 8. therefore since the first Aq is a square number, *d* the third AB shall be a square number also. *Which was to be demonstrated.*

Or thus, Let ab, cd, be like plane numbers; name-
x 19. 7. ly, a. b : : c.d, *x* therefore ad=bc, and so likewise abcd,
y 1. *ax.* 7. or adbc *y* =adad =Q: ad.

PROP. II.

A, 6. B, 54
Aq, 36. AB, 324.

If two numbers A, B, *multiplying one another, produce a square number* AB, *those numbers* A, B, *are like plane numbers.*

a 17. 7. For A. B *a* : : Aq. AB; wherefore since between Aq
b 11. 8. AB, *b* there falls one mean proportional number, *c* like-
c 8. 8. wise one mean shall fall between A and B, *d* therefore
d 20. 8. A and B are like planes. *Which was to be demonstrated.*

PROP. III.

If a cube number Ac *multiplying it self produce a number* Acc, *the number produced* Acc *shall be a cube number.*

A, 2. Ac, 8. Acc, 64.

For, 1. A *a* : : A. Aq *b* : : Aq. Ac, therefore between 1 and Ac fall two mean proportionals. But 1. Ac *a* : : Ac. Acc, *c* therefore between Ac and Acc, fall also two mean proportionals; and so by consequence, since Ac is a cube, *d* Acc shall be a cube also. *Which was to be demonstrated.*

a 15. *def.* 7.
b 17. 7.
c 8. 8.
d 23. 8.

Or thus; aaa (Ac) multiplied into it self makes aaaaaa (Acc;) this is a cube, whose side is aa.

PROP. IV.

If a cube number Ac *multiplying a cube number* Bc *produce a number* AcBc, *the produced number* AcBc *shall be a cube.*

Ac, 8. Bc, 27.
Acc, 64. AcBc, 216.

For Ac. Bc *a* : : Acc. AcBc. But between Ac and Bc *b* two mean proportional numbers fall; *c* therefore there fall as many between Acc and AcBc. So that whereas Acc is a cube number, *d* AcBc shall be such also. *Which was to be demonstrated.*

a 17. 7.
b 12. 8.
c 8. 8.
d 23. 8.

Or thus. AcBc = aaabbb (ababab) = C : ab.

PROP. V.

If a cube number Ac *multiplying a number* B *produce a cube number* AcB, *the number multiplied* B *shall also be a cube.*

Ac, 8. B, 27.
Acc, 64. AcB, 216.

For Acc. AcB *a* : : Ac. B. But between Acc and AcB *b* fall two mean proportionals; *c* therefore also as many shall fall between Ac and B, whence Ac being a cube number, *d* B shall be a cube number also. *Which was to be demonstrated.*

a 17. 7.
b 12. 8.
c 8. 8.
d 23. 8.

PROP. VI.

A, 8. Aq, 64. Ac, 512. *If a number A multiplying it self produce Aq a cube, that number A it self is a cube.*

a hyp. b 19. def. 7. c 5. 9.

For because Aq *a* is a cube, and AqA (Ac) *b* also a cube; therefore *c* shall A be a cube. *Which was to be demonstrated.*

PROP. VII.

A, 6. B, 11. AB, 66.
D, 2. E, 3.

If a composed number A multiplying any number B, produce a number AB, the number produced AB shall be a solid number.

a 13. def. 7. b 9. ax. 7. c 17. def. 7.

Since A is a composed number, *a* some other number D measures it, conceive by E, *b* therefore A = DE: *c* whence DEB = AB is a solid number. *Which was to be demonstrated.*

PROP. VIII.

1. a, 3. a^2, 9. a^3, 27. a^4, 81. a^5, 242. a^6, 729.

If from unity there are numbers continually proportional how many soever (1. a, a^2, a^3, a^4, &c) *the third number from unity* a^2 *is a square number; and so are all forward, leaving one between* (a^4, a^6, a^8, &c.) *But the fourth* a^3 *is a cube number; and so are all forward, leaving two between* (a^6, a^9, &c) *The seventh also* a^6 *is both a cube number and a square; and so are all forward, leaving five between* (a^{12}, a^{18}, &c)

For 1. a^2 = Q. a, and a^4 = aaaa = Q. aa, and a^6 = aaaaaa = Q. aaa, &c.

2. a^3 = aaa = C. a, and a^6 = aaaaaa = C. aa, and aaaaaaaaa = C. aaa, &c.

3. a^6 = aaaaaa = C. aa = Q. aaa, therefore, &c.

a hyp. b 20. 7. c 12. 8. d 23. 8.

Or according to *Euclide*; Because 1. a *a* : : a. a^2, *b* shall a^2 = Q : a, therefore seeing a^2, a^3, a^4, are ∺, *c* the third a^4 shall be a square number; and so likewise a^6, a^8, &c. Also because 1. a *a* : : a^2. a^3, therefore shall a^3 *b* = a^2 x a = C : a, *d* therefore the fourth from a^3, namely a^6, shall be likewise a cube, &c. and consequently a^6 is both a cube and a square number, &c.

PROP.

PROP. IX.

If from unity, $1.\ a,\ 4.\ a^2,\ 16.\ a^3,\ 64.\ a^4,\ 256,$ &c.
there are numbers $1.\ a,\ 8.\ a^2,\ 64.\ a^3,\ 512.\ a^4,\ 4096.$
how many soever,
continually proportional ($1:\ a^2,\ a^3,$ *&c.*) *and the number following unity,* (a) *be a square; then all the rest,* $a^2,\ a^3,\ a^4,$ &c. *shall be squares also. But if the number next unity* (a) *be a cube, then all the following numbers* $a^2,\ a^3,\ a^4,$ &c. *shall be cube numbers.*

1. *Hyp.* For $a^2,\ a^4,\ a^6,$ *&c.* are square numbers by the preceding prop. also since a is taken to be a square, *a* therefore the third a^3 shall be a square, and likewise $a^5,\ a^7,$ *&c.* and so all. a 22. 8.

2. *Hyp.* a is put a cube, *b* therefore $a^4,\ a^7,\ a^{10}$ are cubes; but by the prec. $a^3,\ a^6,\ a^9,$ *&c.* are cubes: lastly, because $1.\ a::a.\ aa,$ *c* therefore shall $a^2 = Q:a,$ but a cube multiplied into it self *d* produces a cube; therefore a^2 is a cube, *e* and consequently the fourth from it a^5; and in like manner $a^8,\ a^{11},$ *&c.* are cubes, therefore all, *&c.* *Which was to be demonstrated* b 23. 8. c 20. 7. d 3. 9. e 23. 8.

Peradventure more clearly thus. Let b be the side of the square number a, and so the series $a,\ a^2,\ a^3,\ a^4,$ *&c.* will be otherwise expressed, thus, $bb,\ b^4,\ b^6,\ b^8,$ *&c.* It is evident that all these numbers are squares, and may be thus expressed, $Q:b,\ Q:bb:Q:bbb,\ Q:bbbb,$ *&c.*

In like manner, if b be the side of the cube a, the series may be expressed thus, $b^3,\ b^6,\ b^9,\ b^{12},$ *&c.* or $C:b,\ C:b^2,\ C:b^3,\ C:b^4,$ *&c.*

PROP. X.

If from unity, there are $1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,$
numbers how many soever con- $1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64.$
tinually proportional ($1.\ a,\ a^2,$ $a^3,$ *&c.*) *and the number next the unit* (a) *be not a square number; then is none of the rest following a square number, excepting* a^2 *the third from unity, and so all forward, leaving one between* ($a^4,\ a^6,\ a^8,$ *&c.*) *But if that* (a) *which is next after the unit, be not a cube number, neither is any other of the following numbers a cube, saving* a^3, *the fourth from the unit, and so all forward, leaving two between,* $a^6,\ a^9,\ a^{12},$ *&c.*

a *hyp.* 1. *Hyp.* For if it be possible, let a^5 be a square number; therefore because $a. a^2 \; a :: a^4. a^5$, and by inver-
b *suppos.* & sion, $a^5. a^4 :: a^2, a$; and also a^5 and a^4 *b* square
8 9. numbers, and the first a^2 a square, *c* therefore a shall
c 24. 8. be likewise a square; *contrary to the hyp.*

d 14. 7. 2. *Hyp.* If it may be, let a^4 be a cube; since *d* by equality $a^4. a^6 :: a. a^3$, and inversely $a^6. a^4 :: a^3. a$; and also since a^6 and a^4 are cubes, and the first a^3 a
e 25. 8. cube, *e* therefore a shall be a cube also; *against the hypothesis.*

PROP. XI.

$1. a, a^2, a^3, a^4, a^5, a^6.$
$1, 3, 9, 27, 81, 243, 729.$

If there are numbers how many soever in continual proportion from unity (1, a, a^2, a^3, *&c.*) *the less measureth the greater by some one of them that are amongst the proportional numbers.*

a 5. *ax.* 7. & 20. *def.* 7. Because $1. a :: a. aa$, *a* therefore $\frac{aa}{a} = a = \frac{aaa}{aa}$. Also
b 14. 7. because $1. aa \; b :: a. aaa$, *a* therefore $\frac{aaa}{a} = aa = \frac{a^4}{aa} = \frac{a^5}{a^3}$, *&c.* Lastly because $1. a^3 :: b \; a. a^4$, therefore $\frac{a^4}{a} = a^3 = \frac{a^6}{a^3}$, &c.

Coroll.

Hence, If a number that measures any one of proportional numbers, be not one of the said numbers, neither shall the number by which it measures the said proportional numbers, be one of them.

PROP. XII.

If there are numbers how many soever in continual proportion from unity ($1, a, a^2, a^3, a^4$.) *whatsoever prime numbers* B *measure the last* a^4, *the same* (B) *shall also measure the number* (a) *which follow next after unity.*

$1, a, a^2, a^3, a^4.$
$1, 6, 36, 216, 1296.$
B, 3.

a 31. 7. If you say B does not measure a, *a* then B is prime to a;
b 27. 7. *b* and therefore B is prime to a^2; *c* and so consequently to
c 26. 7. a^4, which it is supposed to measure. *Which is absurd.*

Coroll.

Coroll.

1. Therefore every prime number that measures the last, does also measure all those other numbers that precede the last.

2. If any number not measuring that next to unity, does yet measure the last, it is a composed number.

3. If the number next to the nuit be a prime, no other pime number shall measure the last.

PROP. XIII.

If from unity there are numbers in continual proportion, how many soever (1, a, a^2, a^3, &c.) *and that after unity* (a) *a prime; then shall no other measure the greatest number, but those which are amongst the said proportional numbers.*

1,	a,	a^2,	a^3,	a^4,
1,	5,	25,	125,	625.
H --	G --	F --	E --	

If it be possible, let some other E measure a^4, *viz.* by F, *a* then F shall be some other different from a, a^2, a^3. But because E measuring a^4, does not measure a, *b* therefore E shall be a composed number, *c* therefore some prime number measures it, *d* which does consequently measure a^4, *e* and so is no other than a, therefore a measures E. After the same manner also may F be shewn to be a composed number, measuring a^4, and so that a measures F. Therefore seeing EF *f* $= a^4$. $= a \times a^3$, *g* shall a. E : : F. a^3. Consequently, whereas a measures E, *h* likewise F shall equally measure a^3, *viz.* by the same number G. *k* Nor shall G be a, or a^2, therefore, as before, G is a composed number, and a measures it. Wherefore since FG *f* $= a^3 = a^2 \times a$, *g* shall a. F :: G. a^2, and so because A measures F, *h* G shall equally measure a^2, *viz.* by the same number H, *k* which is not a. Therefore since GH $= a^2 =$ aa, *l* thence H. a : : a . G, and because a measures G (as before) *m* H also shall measure a, which is a prime number. *Which is impossible.*

a *cor.* 12.9.
b 2. *cor.* 12. 9.
c 33. 7.
d 11 *ax.* 7.
e 3. *cor.* 12. & 9.
f 9. *ax.* 7.
g 19. 7.
h 20. *def.* 7.
k *cor.* 11.9.
l 20. 7.
m 20 *def.* 7.

PROP. XIV.

A, 30.
B, 2. C, 3. D, 5.
E - - F - - -

If certain prime numbers B, C, D, *do measure the least number* A, *no other prime number* E *shall measure the same, besides those that measured it at first.*

a 9. ax. 7. b 32. 7. If it is possible, let $\frac{A}{E}$ be $=$ F, *a* then A $=$ EF, *b* therefore every of the prime numbers B, C, D, measures one of those E, F. Not E, which is taken to be a prime; therefore F, which is less than A it self; *contrary to the hypothesis.*

PROP. XV.

A, 9. B, 12. C, 16.
D, 3. E, 4.

If three numbers continually proportional A, B, C, *are the least of all that have the same proportion with them; any two of them added together shall be a prime to the third.*

a 35. 7. b 2. 8. c 24. 7. d 30. 7. * 26. 7. e 3. 2. f before g 27. 7. h 26. 7. k 4. 2. l 30. 7.

a Take D and E the least in proportion of A to B; *b* then A $=$ Dq, and *b* C $=$ Eq, *b* and B $=$ DE. But because D *c* is prime to E, *d* therefore shall D $+$ E be prime to both D and E, * therefore D x D $+$ E *e* $=$ Dq $+$ DE (*f* A $+$ B) is prime to E, and so to C or Eq. *Which was to be demonstrated.*

g In like manner DE $+$ Eq (B $+$ C) is prime to D, and consequently to A $=$ Dq. *Which was to be demonstrated.*

Lastly, because B *h* is prime to D $+$ E, it shall also be prime to the square of it *k* Dq $+$ 2 DE $+$ Eq (A $+$ 2 B $+$ C;) *l* wherefore the said B shall be prime to A $+$ B $+$ C, *l* and so likewise to A $+$ C. *Which was to be demonstrated.*

PROP. XVI.

If two numbers A, B, *are prime to one another, it shall not be as the first* A, *to the second* B, *so is the second* B *to any other* C.

A, 3. B, 5. C - - - -

If

If you affirm A. B : : B. C, then whereas A and B *a* are the least in their proportion, A *b* shall measure B as many times as B does C; but A *c* measures it self also; therefore A and B are not prime to one another. *against the hypothesis.* a 23. 7. b 21. 7. c 6. *ax*. 7.

PROP. XVII.

A, 8. B, 12. C, 18. D, 27. E - - ?

If there are numbers how many soever in continual proportion A, B, C, D, *and the extremes of them* A, D, *be prime one to another, the first* A *shall not be to the second* B, *as the last* D *to any other* E.

Suppose A. B : : D. E, then alternately A. D : : B. E, therefore seeing A and B are *a* the least in their proportion, A *b* shall measure B, *c* and B likewise C, and C the following number D, *d* and so A shall measure the said number D. Wherefore A and D are not prime to one another; *contrary to the hypothesis.* a 23. 7. b 21. 7. c 20. *def*. 7. d 11. *ax*. 7.

PROP. XVIII.

A, 4. B, 6. C, 9.
Bq, 36.

Two numbers being given A, B, *to consider if there may be a third number* C *found proportional to them.*

If A measures Bq by any number C, *a* then AC = Bq, from whence *b* it is manifest that A. B : : B. C. *Which was to be done.* a 9. *ax*. 7. b 20. 7.

A, 6. B, 4. Bq 16.

But if A does not measure Bq, there will not be any third proportional. For suppose A. B : : B. C, *a* then AC = Bq, *c* and consequently $\frac{Bq}{A}$ = C, namely A measures Bq. *Which is against the Hypothesis.* c 7. *ax*. 7.

PROP. XIX.

A, 8. B, 12. C, 18. D, 27.
BC, 216.

Three numbers being given A, B, C, *to consider if a fourth proportional to them* D *may be found.*

If A measures BC by any number D, *a* then AD = BC; *b* therefore it appears that A. B : : C. D, which was required. a 9 *ax*. 7. b *ax*. 19. 7.

But if A does not meaſure BC, then there can no fourth proportional be found; which may be ſhewn as in the prec. *prop.*

PROP. XX.

A, 2. B, 3. C, 5.
D, 30. G----

More prime numbers may be given than any multitude whatſoever of prime numbers A, B, C, *propounded.*

a 38. 7. b 33. 7. c *ſuppoſ.* d *conſtr.* e 12. *ax.* 7.

a Let D be the leaſt which A, B, C, meaſure; If D + 1 be a prime, the caſe is plain; if compoſed, *b* then ſome prime number, ſuppoſe G, meaſures D + 1, which is none of the three A, B, C; For if it be, ſeeing it *c* meaſures the whole D+1, *d* and the part taken away D, *e* it ſhall alſo meaſure the remaining unit. *Which is abſurd.* Therefore the propounded number of prime numbers is increaſed by D + 1, or at leaſt by G.

PROP. XXI.

5 5 3 3 2 2
A.....E.....B...F...C..G..D 20.

If even numbers, how many ſoever: AB, BC, CD, *are added together, the whole* AD *ſhall be even.*

a 6 *def.* 7. b 12. 7. c 6. *def.* 7.

a Take EB = $\frac{1}{2}$ AB, and FC = $\frac{1}{2}$ BC, and GD = $\frac{1}{2}$ CD, *b* it is plain that EB + FC + GD = $\frac{1}{2}$ AD, *c* therefore AD is an even number. *Which was to be demonſtrated.*

PROP. XXII.

1 1 1
A.......F.B......G.C....H.D..L.E 22
9 7 5 3

If odd numbers, how many ſoever, AB, BC, CD, DE, *are added together, and the multitude of them be even, the whole alſo* AE *ſhall be even.*

a 7. *def.* 7. b 21. 9. c *hyp.* d 21. 9.

Unity being taken from each odd number, there will *a* remain AF, BG, CH, DL, even numbers, *b* and thence the number compounded of them will be even, add to them the *c* even number made of the remaining units, and the *d* whole AE will thereby be even. *Which was to be demonſtrated.*

PROP.

PROP. XXIII.

7 5 1
A.......B.....C..E.D 15.
3

If odd numbers how many soever, AB, BC, CD, *are added together, and the multitude of them be odd, the whole* AD *shall be odd.*

For CD one of the odd numbers being taken away, the aggregate of the others A C *a* is even. Whereto add CD − 1, *b* the whole AE is also even; wherefore the unit being restored the whole AD *c* will be odd. *Which was to be demonstrated.*

a 22. 9.
b 21 9.
c 7. *def.* 7.

PROP. XXIV.

4 5 1
A.... B,.....D. C 10.
6

If an even number AB *be taken away from an even number* AC, *that which remains* BC *shall be even.*

For if BD (BC − 1) be odd, *a* BC (BD + 1) will be even. *Which was to be dem.* But if you say BD is even, because AC *b* is even, *c* thence AD will be so; *a* and consequently AC (AD + 1) will be odd, *contrary the Hypothesis*, therefore BC is even. *Which was to be demonstrated.*

a 7 *def.* 7.
b *hyp.*
c 21. 9.

PROP. XXV.

If from an even number AB, *an odd number* AC *be taken away, the remaining number* CB *shall be odd.*

6 1 3
A......D.C...B 10.
7

For AC − 1 (AD) *a* is even, *b* therefore DB is even; *c* and consequently CB (DB − 1) is odd. *Which was to be demostrated.*

a 7. *def.* 7.
b 24 9.
c 7. *def.* 7.

PROP. XXVI.

If from an odd number AB *be taken away an odd number* CB, *that which remaineth* AC *shall be even.*

4 6 1
A....C.......D. B 11.

a 7. def. 7. b 24. 9.

For AB — 1 (AD) and CB — 1 (CD) *a* are even; *b* therefore AD — CD (AC) is even. *Which was to be demonstrated.*

PROP. XXVII.

A . D . . . . C . . . . . . B 11 (AD 1, DC 4, CB 6, AC 5)

If from an odd number AB *be taken away an even number* CB, *the residue* AC *shall be odd.*

a 7. def. 7. b 24. 9. c 7. def. 7.

For AB — 1 (DB) *a* is even, and CB is supposed to be even; *b* therefore the residue CD is even: *c* therefore CD + 1 (CA) is odd. *Which was to be demonstrated.*

PROP. XXVIII.

A, 3
B, 4
AB, 12.

If an odd number A, *multiplying an even number* B, *produces a number* AB, *the number produced* AB *shall be even.*

a hyp. and 15 def. 7. b 21. 9.

For AB *a* is compounded of the odd number A taken as many times as an unite is contained in B an even number. *b* Therefore AB is an even number.

Schol.

In like manner, if A be an even number, AB shall be be an even number also.

PROP. XXIX.

A, 3
B, 5
AB, 15.

If an odd number A *multiplying an odd number* B, *produces a number* AB, *the number produced* AB, *shall be odd.*

a 15. def. 7. b 23. 9.

For AB *a* is compounded of the odd number B taken as often as an unit is included in A likewise an odd number. *b* Therefore AB is an odd number. *Which was to be demonstrated.*

Schol.

B, 12. (C, 4.
A, 3.

1. *An odd number* A *measuring an even number* B, *measures the same by an even number* C.

a 9. ax. 7. b 29 9.

For if C be affirmed to be odd, then because *a* B = AC, *b* therefore B shall be odd; *against the hyp.*

2. An

2. *An odd number* A *measuring an odd number* B, *measures the same by an odd number* C. $\frac{B, 15.}{A, 3.}$ (C, 5.

For if C be said to be even, *a* then AC, or B will be even, *contrary to the Hypothesis.* a 28. 9.

3. *Every number* (A *and* C) *that measures an odd number* B, *is it self an odd number.* $\frac{B, 15.}{A\ \ 3.}$ C, 5.

For if either A or C be affirmed to be even, B *a* shall be an even number, *against the Hypothesis.* a 28. 9.

PROP. XXX.

$\frac{B, 24.}{A, 3.}$ (C, 8. $\frac{D, 12.}{A, 3.}$ (E, 4.

If an odd number A *measures an even number* B, *it shall also measure the half of it* D.

a Let $\frac{B}{A}$ be $=$ C, *b* then C is an even number. Therefore let E be $= \frac{1}{2}$ C; then B *c* $=$ CA *d* $=$ 2 EA *e* $=$ 2 D, *f* therefore EA $=$ D; *g* and consequently $\frac{D}{A} =$ E. *Which was to be demonstrated.*

a *hyp.*
b 1 *schol.* 29. 9.
c 9. *ax.* 7.
d 1. 2.
e *hyp.*
f 7. *ax.* 1.
g 7. *ax.* 7.

PROP. XXXI.

If an odd number A *be prime to any number* B, *it shall also be prime to the double thereof* C. A, 5. B, 8. C, 16. D---

If it be possible, let some number D measure A and C, *a* then D measuring the odd number A shall be odd it self, *b* and so shall measure B the half of the even number C, therefore A and B are not prime one to another. *Which is against the Hypothesis.*

a 3. *schol.* 29. 9.
b 30. 9.

Coroll.

It follows from hence, that an odd number which is prime to any number of a double progression, is also prime to all the numbers of that progression.

PROP.

PROP. XXXII.

1. A, 2. B 4. C, 8. D, 16. *All numbers* A, B, C, D, &c. *in double progression from two, are evenly even only.*

a 6. def. 7. b 20 def 7. c 11. 9 d 8. def. 7. e 13 9.

It is evident that all these numbers A, B, C, D, *a* are even, and *b* ÷:, namely in a double proportion, *c* and so every less measures the greater by some one of them. *d* Wherefore all are evenly even. But because A is a prime number, *e* no number beside these shall measure any of them. Therefore they are evenly even only. *Which was to be demonstrated.*

PROP. XXXIII.

A, 30. B, 15. D - - - E - - *If of a number* A, *the half* B *be odd, the same* A *is evenly odd only.*

a hyp. b 9. def 7. c 8. def. 7. d 9. ax. 7. e 19. 7. f 6 def. 7. g 20. def. 7.

Since an odd number B *a* measures A by two an even number, *b* therefore B is evenly odd. If you affirm it to be evenly even, *c* then some even number D measures it by an even number E, whence 2 B *d* = A *d* = DE; *e* therefore 2. E : : D. B, and therefore as 2 *f* measures the even number E, *g* so D an even number measures B an odd. *Which is impossible.*

PROP. XXXIV.

A, 24. *If an even number* A *be neither doubled from two, nor have it's half part odd, it is both evenly even and evenly odd.*

a 7. def. 7. b 1. sch. 29. 9.

It is evident that A is evenly even, because the half of it is not odd. But because, if A be divided into two equal parts, and the half of it be again divided into two equal parts, and if this be always done, we shall at length fall upon some *a* odd number, (for we cannot fall upon the number two, because A is not supposed to be doubled upward from two) which shall measure A by an even number; for (*b* otherwise A it self should be odd, *against the Hyp*) Therefore A is evenly odd. *Which was to be demonstrated.*

PROP. XXXV.

A........ 8.
4 8
B....F........G 12.
C................. 18.
9 6 4 8
D.........H......L....K........N 27.

If there are numbers in continual proportion how many soever A, BG, C, DN, *and the number* FG, *equal to the first* A, *be taken from the second, and* KN *also equal to the first, from the last; it shall be as the excess of the second* BF *is to the first* A, *so is the excess of the last* DK *to all the numbers that precede it,* A, BG, C.

From DN take NL = BG, and NH = C. Because DN. C (HN) *a* :: HN. BG (LN) *a* : : LN (BG.) A (KN.) *b* therefore by dividing every where, shall DH. HN : : HL. LN : : LK. KN, *c* wherefore DK. C + BG + A : : LK (*d* BF.) . KN (A.) *Which was to be demonstrated.* a *hyp.* b 17. 5. c 12. 5. d 3. *ax.* 1.

Coroll.

Hence *e* by compounding, DN + BG + C. A + BG + C : : BG. A. e 18. 5.

PROP. XXXVI.

1. A, 2. B, 4. C, 8. D, 16.
E, 31. G, 62. H, 124. L, 248. F, 496.
M, 31. N, 465.
P---- Q---

If from unity be taken how many numbers soever 1, A, B, C, D, *in double proportion continually, until the whole added together* E *be a prime number; and if this whole* E *multiply'd into the last* D, *produce a number* F, *that which is produced* F, *shall be a perfect number.*

Take as many numbers E, G, H, L, likewise in double proportion continually; then *a* by equality A. D : : E. L, *b* therefore A L = D E *c* = F, *d* whence L = $\frac{F}{2}$. Wherefore E, G, H, L, F, are ∺ in double proportion. Let G − E be = M, and F − E = N; *e* then M. E : : N. E + G + H + L. *f* But M = E, *g* there- a 14. 7. b 19. 7. c *hyp.* d 7. *ax.* 7. e 35. 9. f 3. *ax.* 1. g 14. 5.

h 2. *ax.* 1. therefore $N=E+G+H+L$, *h* therefore $F = 1 + A +$
k 7. *ax.* 7. $+ B + C + D + E + G + H + L = E + N$. More-
l 11. *ax* 7. over because D *k* measures DE (F) *l* therefore every one,
m 11. 9. 1, A, B, C, *m* measuring D, and *m* also E, G, H, L,
do measure F. And further, no other number measures
the said F. For if there does, let it be P, which let
n 9. *ax.* 7. measure F by Q, *n* therefore $PQ=F=DE$, *o* therefore
o 19. 7. E. Q :: P. D, therefore seeing A a prime number mea-
p 13. 9. sures D, *p* and so no other P measures the same, *q* conse-
q 20. *def.* 7. quently E does not measure Q. Wherefore since E is
r 31. 7. supposed a prime number, *r* it shall be prime to Q, *s*
s 23. 7. wherefore E and Q are the least in their proportion; *t*
t 21. 7. and so E measures P as many times as Q does D, *u* there-
u 13 9. fore Q is one of them A, B, C. Let it be B, seeing
x 19. 7. then by equality B. D :: E. H, *x* and so $BH=DE=F$
y 14. 5. $=PQ$, *x* and so also Q. B :: H. P, *y* therefore $H = P$,
therefore P is also one of them A, B, C, &c. *Against the Hopothesis.* Therefore no other beside the foresaid
numbers measures F, and *z* consequently F is a perfect
z 22. *def.* 7. number. *Which was to be demonstrated.*

The end of the ninth Book.

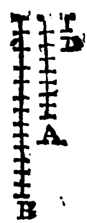
THE

The Tenth Book OF EUCLIDE'S ELEMENTS.

Definitions.

I. MAgnitudes are said to be commensurable which one and the same measure measures.

The note of commensurability is ⊐⊏, *as* A ⊐⊏ B; *that is, the line* A *of* 8 *foot is commensurable to the line* B *of* 13 *foot; because* D *a line of one foot measures both* A *and* B. *Also* $\sqrt{18}$ ⊐⊏ $\sqrt{50}$; *because* $\sqrt{2}$ *measures both* $\sqrt{18}$ *and* $\sqrt{50}$. *For* $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$, *and* $\sqrt{\frac{50}{2}} = \sqrt{25} = 5$, *wherefore* $\sqrt{18} . \sqrt{50} :: 3 . 5$.

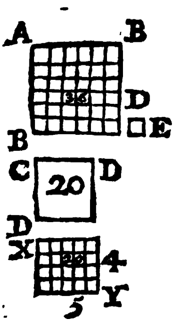


II. Incommensurable magnitudes are such, of which no common measure can be found.

Incommensurability is denoted by this mark ⊐⊢; *as* $\sqrt{6}$ ⊐⊢ $\sqrt{25}$ (5;) *that is,* $\sqrt{6}$ *is incommensurable to the number* 5, *or to a magnitude designed by that number; because there is no common measure of them, as shall appear hereafter*

III. Right lines are commensurable in power, when the same space does measure their squares.

The



The mark of this commensurability is ⊐; *as* AB ⊐ CD, *i.e. the line* AB *of 6 foot is in power commensurable to the line* CD, *which is expressed by* $\surd$ 20, *because* E *the space of one foot square does as well measure* ABq (36) *as the rectangle* XY (20) *to which the square of the line* CD ($\surd$ 20) *is equal. The same note* ⊐ *sometimes signifies commeasurable in power only.*

IV. Lines incommensurable in power are such, to whose squares no space can be found to be a common measure.

This incommensurability is denoted thus; 5 ⊐ $v \surd 8$. *i. e. the numbers or lines* 5, *and* $v \surd$ 8 *are incommensurable in power, because their squares* 25 *and* $\surd$ 8 *are incommensurable.*

V. From which it is manifest, that to any right line given, right lines infinite in multitude are both commensurable and incommensurable; some in length and power, others in power only. The right line given is called a Rational line.

The note of which is $\acute{\rho}$.

VI. And lines commensurable to this line, whether in length and power, or in power only, are also called Rational, $\acute{\rho}$.

VII. But such as are incommensurable to it, are called Irrational,

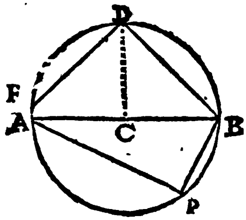
And denoted thus $\dot{\rho}$.

VIII. Also the square which is made of the said given right line is called Rational, $\acute{\rho}\nu$.

IX. And likewise such figures as are commensurable to it, are Rational $\acute{\rho}\alpha$.

X. But such as are incommensurable, Irrational $\dot{\rho}\alpha$.

XI. And those right lines also, which contain them in power, are Irrational $\dot{\rho}$.



Schol.

That the last seven definitions may be rendered more clear by an example, let there be a circle ADBP, *whose semidiameter is* CB, *inscribe therein the sides of the ordinate figures, as of a Hexagone* BP, *of a triangle* AP, *of a square* BD, *of a Pentagone*

gone FD. *Therefore, if according to the* 5. def. *the semidiameter* CB *be the Rational line given, expressed by the number* 2, *to which the other lines* BP, AP, BD, FD, *are to be compared, then* BP a = BC = 2, *wherefore* BP *is* ῥ ⊐ BC, a cor. 15. 4.
according to the 6 def. *Also* AP b = √ 12 (*for* ABq (16) b 47. 1.
— BPq (4) = 12) *therefore* AP *is* ῥ ⊐ BC *likewise according to the* 6. def. *and* APq (12) *is* ῥν *by the* 9. def. *Moreover* BD b = √ DCq + BCq = 8; *whence* BD *is* ῥ ⊐ BC; *and* BDq ῥν. *Lastly*, FDq = 10 — √ 20 (*as shall appear by the praxis to be delivered at the* 10. 13.) *shall be* ῥα, *according to the* 10. def. *and consequently* FD = √ : 10 — √ 20 *is* ῥ, *according to the* 11. def.

A Postulate.

That any magnitude may be so often multiplyed, till it exceed any magnitude whatsoever of the same kind.

Axioms.

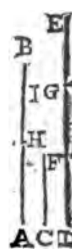
1. A magnitude measuring how many magnitudes soever, does also measure that which is composed of them.

2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.

3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue.

PROP. I.

Two unequal magnitudes AB, C, *being given, if from the greater* AB *there be taken away more than half* (AH) *and from the residue* (HB) *be again taken away more than half* (HI) *and this be done continually, there shall at length be left a certain magnitude* IB, *less than the lesser of the magnitudes first given* C.



a Take C so often, till its multiple does somewhat exceed AB, and let DF = FG = GE = C. Take from AB more than half AH, and from the remainder HB, more than half *viz.* HI, and so continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE which is not less than ½ DE, is greater than HB, which is less than ½ AB ⊐ DE. And in like manner GE, which is not less than ½ FE, is greater than IB ⊐ ½ HB, therefore C, or GE ⊏ IB. *Which was to be demonstrated.* a post. 10.

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.

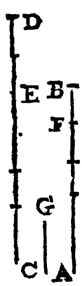
PROP.

PROP. II.

Two unequal magnitudes being given (AB, CD) *if the less* AB *be continually taken from the greater* CD, *by an interchangeable subtraction, and the residue do not measure the magnitude going before, then are the magnitudes given incommensurable.*

If it be possible, let some magnitude E be the common measure. Then because AB taken from CD, as often as it can be, leaves a magnitude FD less than it self, and FD taken from AB leaves GB, and so forward
a 1. 10. *a* therefore at length some magnitude GB
b *hyp.* $\sqsubset$ E shall be left, therefore E *b* measuring AB, *c* and
c 2. *ax.* 10. so CF, *b* and the whole CD, *d* shall also measure the
d 3. *ax.* 10. residue FD, *c* consequently also AG; *d* wherefore it shall likewise measure the remainder GB, less than it self. *Which is absurd.*

PROP. III.



Two commensurable magnitudes being given AB, CD, *to find out their greatest common measure* FB.

Take AB from CD, and the residue ED from AB, and FB from ED, till FB measure ED (which will come to pass at length, *a* because by the Hyp. AB $\sqcap$ CD)
a 2. 10. FB shall be the magnitude required.

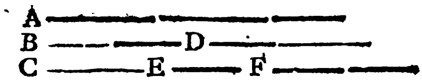
For FB *b* measures ED, *c* and so also
b *constr.* AF; but it measures it self too, *d* there-
c 2. *ax* 10. fore likewise AB, *c* and consequently CE,
d 1. *ax.* 10. *d* and so the whole CD. Wherefore FB is
e 2. *ax.* 10. the common measure of AB, CD. If you affirm G to be
f 3. *ax.* 10. a common measure greater than that, then G measuring AB and CD, *e* measures also CE and *f* the remainder ED, *e* and so AF; and *f* consequently the remainder FB, the greater the less. *Which is absurd.*

Coroll.

Hence, a magnitude that measures two magnitudes, does also measure their greatest common measure.

PROP.

PROP. IV.



Three commensurable magnitudes being given, A, B, C, *to find out their greatest common measure.*

a Find out D the greatest common measure of any two A, B; *a* also E the greatest common measure of D and C, therefore E is the magnitude sought for. a 3. 10.

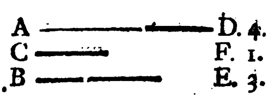
a For it is clear, E measuring D and C, *b* does measure the three, A, B, C. Conceive another magnitude F greater than that to measure them; *c* then F measures D, *c* and consequently E the greatest common measure of D, and C, the greater the less. *Which is absurd* b *constr.* & 2 *ax.* 10. c *cor.* 3. 10.

Coroll.

Hence also it appears, that if a magnitude measures three magnitudes, it shall likewise measure their greatest common measure.

PROP. V.

Commensurable magnitudes A, B, *have such proportion one to another, as number hath to number.*



a C being found the greatest common measure of A, B; as often as C is contained in A and B, so often is 1 contained in the numbers D and E; *b* therefore C. A : : 1. D; wherefore inversly A. C : : D. 1, *b* but likewise C. B : : 1. E; *c* therefore by equality A. B : : D. E N. N. *Note,* The letter N only signifies number in general, and refers not to any particular space or magnitude as the other letters do, and is to be read, as A to B, so D to E, and so number to number. *Which was to be demonstrated.* a 3. 10. b 20. *def.* 7. c 22. 5.

M PROP.

PROP. VI.

E —— F, 1.
A ———— C, 4.
B ——— D, 3.

If two magnitudes A, B, *have such proportion one to another, as the number* C *hath to the number* D, *those magnitudes* A, B, *shall be commensurable*

a *sch.* 10. 6. What part 1 is of the number C, *a* that let E be of
b *constr.* A. Therefore because E. A *b* :: 1. C, and A. B *c* ::
c *hyp.* C. D, *d* therefore by equality shall E. B :: 1. D. Where-
d 22. 5. fore seeing 1 *e* measures the number D, *f* likewise E mea-
e 5. *ax.* 7. sures B; but it *g* also measures A, *h* therefore A ⊐⊏ B.
f 20 *def.* 7. *Which was to be demonstrated.*
g *constr.*
h 1. *def.* 10.

PROP. VII.

A ————
B ————

Incommensurable magnitudes A, B, *have not that proportion one to another, which number hath to number.*

a 6. 10. If you affirm A. B :: N. N, *a* then A ⊐⊏ B, *against the Hypothesis.*

PROP. VIII.

A ————
B ————

If two magnitudes A, B, *have not that proportion one to another, which number hath to number, those magnitudes are incommensurable.*

a 5. 10. Conceive A ⊐⊏ B, *a* then A. B :: N. N, *contrary to the Hypothesis.*

PROP. IX.

A ————
B ——
E, 4.
F, 3.

The squares described upon right lines commensurable in length, have that proportion one to another, that a square number hath to a square number. And squares, which have that proportion one to another, that a square number hath to a square number, shall also have their sides commensurable in length. But such squares as are made upon right lines incommensurable in length, have not that proportion one to another, which a square number hath to a square number. And squares which have not such proportion one to another, as a square number hath to a square number, have not their sides commensurable in length.

1. *Hyp.*

1. *Hyp* A ⌐⊔ B. I say that Aq. Bq : : Q. Q.

For *a* let A. B : : number E. number F; therefore $\frac{Aq}{Bq}$ (*b* $\frac{A}{B}$ twice) *c* = $\frac{E}{F}$ twice, *d* = $\frac{Eq}{Fq}$, *e* therefore Aq. Bq : : Eq. Fq : : Q. Q. *Which was to be demonstrated*

2. *Hyp.* Aq. Bq : : Eq. Fq : : Q. Q. I say A ⌐⊔ B. For $\frac{A}{B}$ twice (*f* $\frac{Aq}{Bq}$) *g* = $\frac{Eq}{Fq}$ *h* = $\frac{E}{F}$ twice, *i* therefore A. B : : E. F : : N. N, *k* wherefore A ⌐⊔ B. *Which was to be demonstrated.*

3. *Hyp.* A '⌐⊔ B. I deny that Aq. Bq : : Q. Q. For suppose Aq. Bq : : Q. Q, then A ⌐⊔ B, as is shewn before, *against the Hypothesis*

4. *Hyp.* Not Aq. Bq : : Q. Q, I say that A '⌐⊔ B. For conceive A ⌐⊔ B, then Aq. Bq : : Q. Q, as above, *against the Hypothesis.*

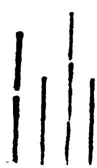
a 5. 10.
b 20 6.
c *sch* 23. 5
d 11. 8.
e 11 5.
f 20. 6.
g *hyp.*
h 11. 8.
i *sch* 23. 5.
k 6. 10.

Coroll.

Lines ⌐⊔ are also ⌐⊔̦, but not on the contrary. And lines '⌐⊔ are not therefore '⌐⊔̦, but Lines '⌐⊔̦ are also '⌐⊔.

PROP. X.

If four magnitudes are proportional (C. A : : B. D) *and the first* C *be commensurable to the second* A, *the third* B *shall be commensurable to the fourth* D. *And if the first* C *be incommensurable to the second* A, *also the third* B *shall be incommensurable to the fourth* D.


C A B D

If C ⌐⊔ A, *a* then C. A : : N. N. *b* : : B. D, *b* therefore B ⌐⊔ D. But if C '⌐⊔ A, *c* then shall not C. A : : N. N : : B. D, *e* wherefore B '⌐⊔ D. *Which was to be demonstrated.*

a 5. 10.
b 6. 10.
c 7. 10.
d 8. 10.

Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like, will satisfy this Lemma, as those numbers which have super-particular, superbipartient, or double proportion; or any two prime numbers, *See Schol.* 27. 8.

Lemma 2.

B, 5. K——1——1——1——1—— M
C, 3. H——1——1—— R
D————

To find out a line HR, *to which a right line given* KM *hath the proportion of two numbers given* B, C.

a *sch.* 10. 6. *a* Divide KM into as many equal parts as there are units in the number B, and let as many of these, as there
b 3. 1. are units in the number C, *b* make the right line HR, it is manifest that KM. HR : : B. C.

Lemma 3.

To find out a line D, *to the square of which the square of a right line given* KM *hath the proportion of two numbers given* B, C.

a 2 *lem.* 10. Make B. C *a* : : KM. HR, and between KM and HR,
10. *b* find a mean proportional D. Therefore KMq. Dq *c*
b 13 6. : : KM. HR *d* : : B. C.
c 20. 6.
d *constr.*

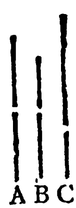
PROP. XI.

A———— B, 20.
E———— C, 16.
D————

To find two right lines incommensurable to a right line given A, *one* D *in length only, the other* E *in power also.*

a 1 *lem.* 10. 1. Take the numbers B, C, *a* so that it be not B. C : :
10. Q. Q, *b* and let B. C : : Aq. Dq, *c* it is plain that A ⌐⊏⊐ D.
b 3 *lem.* 10. But Aq *d* ⊏⊐ Dq. *Which was to be done.*
10. 2. *d* Make A. E : : E. D. I say Aq ⌐⊏⊐ Eq. For A.
c 9. 10. D *e* : : Aq. Eq, therefore since A ⌐⊏⊐ D, as before; *f* there-
d 6 10. fore Aq ⌐⊏⊐ Eq. *Which was to be done.*
d 13. 6.
e 20 6.
f 10. 10.

PROP. XII.



Magnitudes (A, B) *commensurable to the same magnitude* C, *are also commensurable one to the other.*

a 5. 10. Because A ⊏⊐ C, and C ⊏⊐ B, *a* let A. C : : N. N : : D. E, and C. B : : N. N : : F. G, *b* take three
b 4. 8. numbers H, I, K, the least in the proportions of D

D, 8. E, 8.
F, 2. G, 3.
H, 5. I, 4. K, 6.

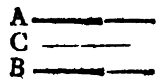
c *constr.* to E, and F to G. Now because A. C *c* : : D. E *c* : : H.
d 22 5. I, and C. B *c* : : F. G : : I. K, *d* therefore by equality,
e 6. 10. A. B : : H. K : : N. N, *e* therefore A ⊏⊐ B. *Which was to be demonstrated.*

Schol.

Schol.

Hence, Every right line commensurable to a rational line is also it self *p* rational. And all rational right-lines are commensurable to one another, at least in power. Also, every space commensurable to a rational space is rational too: And all rational spaces are commensurable one to another. But magnitudes whereof one is rational, the other irrational, are incommensurable amongst themselves. 12. 10. *and def* 6. *def.* 9. *def.* 7. *and* 10.

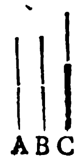
PROP. XIII.



If there are two magnitudes A, B, *and one of them* A, *commensurable to a third* C, *but the other* B *incommensurable, those magnitudes* A B, *are incommensurable.*

Conceive B $\sqcap\!\!\sqcup$ A, then since C *a* $\sqcap\!\!\sqcup$ A, *b* therefore C $\sqcap\!\!\sqcup$ B, *against the hypothesis.* a *hyp.* b 12. 10.

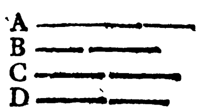
PROP. XIV.



If there are two magnitudes commensurable A, B, *and one of them* A *incommensurable to any other magnitude* C, *the other also* B *shall be incommensurable to the same* C.

Imagine B $\sqcap\!\!\sqcup$ C, then for that A *a* $\sqcap\!\!\sqcup$ B, *b* therefore A $\sqcap\!\!\sqcup$ C, *against the hyp.* a *hyp.* b 12. 10.

PROP. XV.



If four right lines are proportional (A. B : : C. D) *and the first* A *be in power more than the second* B *by the square of a right line commensurable to it self in length, then also the third* C, *shall be more in power than the fourth* D *by the square of a right line commensurable to it self in length. But if the first* A *be more in power than the second* B, *by the square of a right line incommensurable to it self in length, then shall the third* C *be more in power than the fourth* D *by the square of a right line incommensurable to it self in length.*

For because A. B *a* : : C. D, *b* therefore Aq. Bq : : Cq. Dq, *c* therefore by division Aq − Bq. Bq : : Cq − Dq. Dq, *d* wherefore √ : Aq − Bq. B : : √ : Cq − Dq. D, *e* and so inversely B. √ : Aq − Bq : : D √ : Cq − Dq, *f* therefore by equality A. √ : Aq − Bq : : C. √ : Cq − Dq, consequently if A $\sqcap\!\!\sqcup$, or $\sqcap\!\!\sqcup$ √ Aq − Bq, *g* then likewise C $\sqcap\!\!\sqcup$, or $\sqcap\!\!\sqcup$ √ : Cq − Dq. *Which was to be demonstrated.* a *hyp.* b 22. 6. c 17. 5. d 22. 6. e *cor* 4. 5. f 22. 5. g 10. 10.

PROP. XVI.

If two commensurable magnitudes AB, BC, *are composed, the whole magnitude* AC *shall be commensurable to each of the parts* AB, BC. *And if the whole magnitude* AC *be commensurable to either of the parts* AB, *or* BC, *those two magnitudes given at first* AB, BC, *shall be commensurable*,

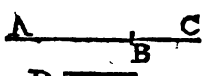
a 3. 10. 1. *Hyp.* *a* Let D be the common measure of AB,
b 1. *ax.* 10. BC; *b* therefore D measures AC, and therefore AC ⊐⊏
c 1. *def.* 10. AB, and BC. *Which was to be demonstrated.*

2. *Hyp.* *a* Let D be the common measure of AC, AB,
d 3. *ax.* 10. *d* thererefore D measures AC — AB (BC) and consequently AB ⊐⊏ BC. *Which was to be demonstrated.*

Coroll.

Hence it follows, if a whole magnitude composed of two, be commensurable to any one of them, the same shall be commensurable to the other also.

PROP. XVII.



If two incommensurable magnitudes AB, BC, *are composed, the whole magnitude also* AC *shall be incommensurable to either of the two parts* AB, BC. *And if the whole magnitude* AC *be incommensurable to one of them* AB, *the magnitudes first given* AB, BC, *shall be incommensurable.*

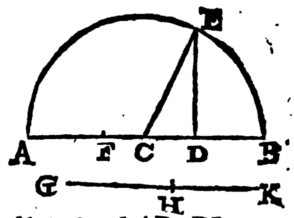
1. *Hyp.* If it can be, let D be the common measure
a 3 *ax.* 10. of AC, AB, *a* therefore D measures AC — AB (BC) *b*
b 1. *def.* 10. and therefore also AB ⊐⊏ BC, *against the hypothesis.*
c 16. 10. 2. *Hyp* Conceive AB ⊐⊏ BC, *c* therefore AC ⊐⊏ AB, *against the hypothesis.*

Coroll.

Hence also, if one magnitude, composed of two, be incommensurable to any one of them, the same also shall be incommensurable to the other.

PROP.

PROP. XVIII.

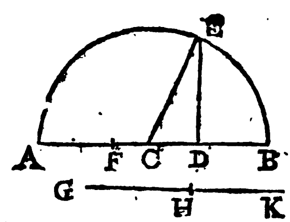


If there are two unequal right lines AB, GK, *and upon the greater* AB *a parallelogram* ADB *equal to the fourth part of a square made of the less line* GK, *and deficient in figure by a square, be applied, and divides the said* AB *into parts commensurable in length* AD, DB; *then shall the greater line* AB *be more in power than the less* GK *by the square of a right line* FD *commensurable in length to the greater. And if the greater* AB *be in power more than the less* GK, *by the square of the right line* FD *commensurable in length to it self, and a parallelogram* ADB *equal to the fourth part of the square made of the less line* GK, *and deficient in figure by a square, be applied to the greater* AB, *then shall it divide the same into parts* AD, DB, *commensurable in length.*

a Divide GK equally in H, and *b* make the rectangle ADB=GHq. Cut off AF=DB, then is ABq *c*=4 ADB *d* (4 GHq or GKq) + FDq. Now in the first place, if AD ⊓⊔ DB, then shall AB *e* ⊓⊔ BD *e* ⊓⊔ 2 DB *f* (AF + DB, or AB − FD) *g* therefore AB ⊓⊔ FD. *Which was to be dem.* But secondly, if AB ⊓⊔ FD, *h* then shall AB ⊓⊔ AB − FD (2 DB) *k* therefore AB ⊓⊔ DB, *l* wherefore AD ⊓⊔ DB. *Which was to be demonstrated.*

a 10. 1.
b 28. 6.
c 8. 2.
d *constr.* & 4. 2.
e 16. 10.
f *constr.*
g *cor.* 16. 10
h *cor.* 16 10.
k 12. 10.
l 16. 10.

PROP. XIX.



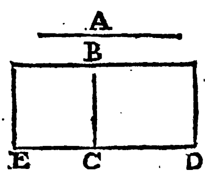
If there are two right lines unequal AB, GK, *and to the greater* AB *a parallelogram* ADB *equal to the fourth part of a square made upon the less* GK, *and deficient in figure by a square be applied, and divides the said* AB, *into parts* AD, DB, *incommensurable in length; the greater line* AB *shall be in power more than the less* GK *by the square of the right line* FD

incommensurable to the greater in length. And if the greater line AB *be more in power than the less* GK *by the square of a right line* FD *incommensurable to it self in length, and if also upon the greater* AB *be applied a parallelogram* ADB *equal to the fourth part of the square of the less* G K, *and deficient in figure by a square, then shall it divide the said greater line* AB, *into parts incommensurable in length* AD, DB.

Suppose all the same that was done and said in the
a 17. 10. prec. prop. Therefore first, If AD $\sqcup$ DB, *a* then shall
b 13. 10. AB $\sqcup$ DB. *b* Wherefore AB $\sqcup$ 2 DB (AB — FD) *c*
c *cor.* 17. 10. therefore AB $\sqcup$ FD. *Which was to be demonstrated.*

Secondly, If A B $\sqcup$ FD, then AB $\sqcup$ AB — F D
d 13. 10. (2 DB) *d* wherefore A B $\sqcup$ DB, *e* and consequently
e 17. 10. AD $\sqcup$ DB. *Which was to be demonstrated.*

PROP. XX.



A rectangle BD *comprehended under right lines* BC, CD, *rational and commensurable in length according to one of the foresaid ways, is rational.*

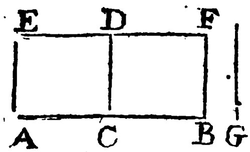
a 46. 1. Let A be given $\acute{\rho}$, and *a* the
b 1. 6. square BE described upon BC.
c *hyp.* Because DC. CE (BC) *b* : : BD. BE, and DC *c* $\sqcup$ BC, *d*
d 10. 10. therefore shall the rectangle BD be $\sqcup$ square BE, where-
e *hyp.* & 9. *def.* 10. fore seeing the square BE *e* $\sqcup$ Aq, shall also *f* BD be $\sqcup$
f 12. 10. Aq, and so the rectangle BD $\acute{\rho}\nu$. *Which was to be demonstrated.*

Note, *There are three kinds of rational lines commensurable one to another. For either of two rational lines commensurable in length one to the other, one is equal to the rational line propounded, or neither of them is equal to it, notwithstanding both of them are commensurable to it in length; or lastly both of them are commensurable to the rational line given only in power. And these are the ways which the present Theorem speaks of.*

In numbers, let there be BC, $\sqrt{8}$ ($2\sqrt{2}$) and CD $\sqrt{18}$ ($3\sqrt{2}$) then shall the rectangle BD = $\sqrt{144} = 12$.

PROP

PROP. XXI.



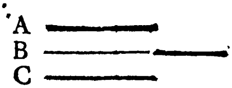
If a rational rectangle DB *be applied to a rational line* DC, *it makes the breadth thereof* CB *rational, and commensurable in length to that line* DC *whereto* DB *is applied.*

Let G be propounded ῥ, and the square DA described on BC, because BD. DA : : *a* BC. CA ; and BD. DA *b* are ῥα *c* and so ⊤⊥. *d* therefore BC ⊤⊥ CA but CD (CA) is ῥ. *e* therefore BC is ῥ. *Which was to be demonstrated.* a 1. 6. b *hyp.* c *sch.* 12 10. d 10 10. e *sch* 12.10.

In numbers, let there be the rectangle DB, 12, and DC, $\sqrt{8}$. then shall CB, $\sqrt{18}$. but $\sqrt{18} = 3\sqrt{2}$. and $\sqrt{8} = 2 \times \sqrt{2}$.

Lemma.

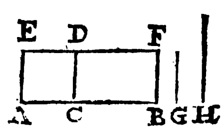
To find out two rational right lines commensurable only in power.



Let A be propounded ῥ. *a* Take B ⊤⊥ A, *a* and C ⊤⊥ B. *b* it is clear that B and C are the lines required. a 11. 10. b *sch.* 12. 10.

PROP. XXII.

A rectangle DB *comprehended under rational right lines* DC, CB *commensurable in power only, is irrational: and the right line* H, *which containeth that rectangle in power is irrational, and called a Medial line.*



Let G be the propounded ῥ, and the square DA described on DC, and let Hq = DB. Because AC. CB *a* : : DA. DB. *b* and AC ⊤⊥ CB, *c* shall be DA ⊤⊥ DB (Hq.) *d* but Gq ⊤⊥ DA. *e* therefore Hq ⊤⊥ Gq *f* wherefore H is ῥ *Which was to be dem.* and let it be called a Medial line, because AC. H : : H. CB a 1. 6. b *hyp.* c 10 10. d *hyp. and* 9 *def* 10. e 13. 10. f 11. 10.

In numbers, let there be DC, 3. and CB, $\sqrt{6}$. then shall the rectangle be DB (Hq) $\sqrt{54}$. wherefore H is $v\sqrt{54}$.

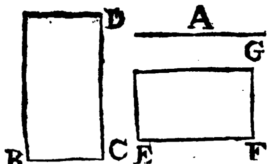
The note of a medial line is μ, of a medial rectangle μν, of more together μα.

Schol.

Schol.

Every rectangle that can be contained under two rational right lines commensurable only in power, is medial, although it be contained under two right lines irrational: and every medial rectangle may be contained under two rational right lines, commensurable only in power; as for example, the $\surd$ 24 is $\mu\nu$, because it is contained under $\surd$ 3, and $\surd$ 8, which are ρ, ⊤⊔ although it may be contained under $v\surd$ 6, and $v\surd$ 96 irrationals; for $\surd 24 = v\surd 576 = v\surd 6 \times v\surd 96$.

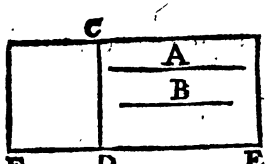
PROP. XXIII.



If the rectangle BD *made of a medial line* A, *be applied to a rational line* BC, *it makes the breadth* CD *rational, and incommensurable in length to the line* BC, *whereunto the rectangle* BD *is applied.*

a *sch.* 22. 10. Because A is μ, *a* therefore shall Aq be equal to some rectangle (EG) contained under EF and FG ρ ⊤⊔. *b*
b 1. *ax.* 1. therefore BD=EG. *d* whence BC.EF :: FG.CD. *d* there-
c 14 6. fore BCq. EFq :: FGq. CDq. But BCq and EFq *e* are
d 22 6. $\rho\alpha$ *f* and so ⌐⊔. *g* therefore FGq ⌐⊔ CDq. Where-
e *hyp.* fore since FG is ρ, *h* therefore CD shall be ρ. Moreover,
f *sch* 12. 10. because EF. FG *k* :: EFq. EG (BD;) for since EF ⌐⊔
g 10. 10. FG, *e* shall EFq be ⌐⊔ BD. But EFq *m* ⌐⊔ CDq. *n*
h *sch.* 12. 10. therefore the rectangle BD ⌐⊔ CDq. Whence since
k 1. 6. CDq. BD *o* :: CD. BC. *p* shall CD be ⌐⊔ BC. there-
l 10. 10. fore, &c.
m *sch.* 12. 10
n 13 10.
o 1. 6.
p 10. 10.

PROP. XXIV.



A right line B *commensurable to a medial line* A, *is also a medial line.*

a 11. 6. Upon CD ρ *a* make the rectangle CE = Aq; *a* and the rectangle CF
b *hyp.* =Bq. Because Aq (CE) is $\mu\nu$, *b* and CD ρ, *c* therefore
c 23. 10. shall the latitude DE be ρ ⊤⊔ CD. But because CE. CF *d* :: ED.

d :: ED. DF. and CE *e* ⊤⊔ CF, *f* therefore ED ⊤⊔ DF. *g* therefore DF is ρ ⊤⊔ CD. *h* whence the rectangle CF (Bq) is $\mu\nu$, and so B is μ, *Which was to be demonstrated.*

Obs. *that the note* ⊤⊒ *for the most part signifies commensurable in power only, as in this and the precedent demonstrations,* &c.

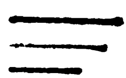
d 1. 6.
c *hyp.*
f 10. 10.
g 12. *and* 13. 10.
h 22. 10.

Coroll.

Hereby it is manifest that a space commensurable to a medial space, is also medial.

Lemma.

To find out two medial right lines A, B, *commensurable in length, and also two,* A, C, *commensurable only in power.*

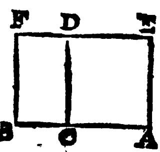
A

B
C

a Let A be any μ, *b* take B ⊤⊔ A, and *c* C ⊤⊒ A, *d* and 'tis evident the thing is done.

a *lem.* 21. 10 *and* 13. 6.
b 2. *lem.* 10. 10.
c 3 *lem.* 10. 10.
d *const. and* 24. 10.

PROP. XXV.

A rectangle DB *contained under* DC, CB *medial right lines commensurable in length, is medial.*

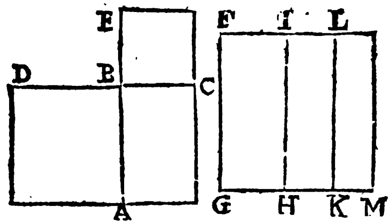


Upon DC describe the square DA. Because AC (DC) CB *a* :: DA. DB, and DC ⊤⊔ CB; *b* shall DA ⊤⊔ DB. *c* therefore DB is $\mu\nu$. *Which was to be demonstrated.*

a 1. 6.
b 10. 10.
c 24. 10.

PROP.

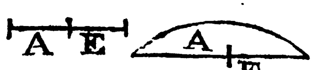
PROP. XXVI.



A rectangle AC *comprehended under medial right lines* AB, BC *commensurable only in power, is either rational or medial.*

a 46. 1. Upon the lines AB, BC, *a* describe the squares AD,
b *cor* 16 6. CE; and upon FG ρ́ *b* make the rectangles FH, = AD,
c *hyp.* & *b* and IK = AC. *b* and LM = CE.
24. 10. The squares AD, CE, that is, the rectangles FH, LM,
d 23. 10. *c* are μα and ⊐. therefore GH, KM, having the same
e 10. 10. proportion *d* are ρ́, *e* and ⊐ *f* therefore GH x KM is
f 20. 10. ρ́ν. But because AD, AC, CE, that is, FH, IK, LM, *g*
g *sch* 22.6. are ∺; *h* and so GH, HK, KM also ∺ ; *k* thence HKq
h 1. 6. = GH x KM. *l* therefore HK is ρ́, or ⊐, or ⊒ IH
k 17. 6. (GF;) if ⊐, *m* then the rectangle IK or AC is ρ́ν. but
l 12. 10. if ⊒, *n* then AC is μν. *Which was to be dem.*
m 20 10.
n 22 10.

Lemma.

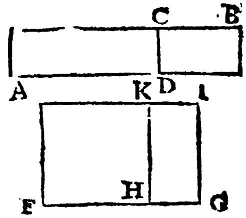


If A *and* E *are* ⊒ *only, Then first,* shall Aq, Eq, Aq + Eq, Aq − Eq
a *hyp and* *a* ⊐. *And secondly* Aq, Eq, Aq + Eq, Aq − Eq ⊐
16. 10. AE and 2 AE. For A. E *b* :: Aq. AE *b* :: AE. Eq.
b 1. 6. therefore seeing A *c* ⊐ E. *d* shall Aq ⊐ AE, *e* and
c *hyp.* 2 AE. also Eq *d* ⊐ AE, *e* and 2 AE. wherefore be-
d 10. 10. cause Aq + Eq ⊐ Aq and Eq; and Aq − Eq ⊐ Aq
e 14. 10. and Eq ; *f* therefore shall Aq + Eq, *f* and Aq − Eq
f 14. 10. be ⊐ AE, and 2 AE.

Hence also thirdly, Aq, Eq, Aq + Eq, Aq − Eq, 2 AE *g* ⊐ Aq + Eq + 2 AE; and Aq + Eq − 2
g 14. 10. AE. *g* and Aq + Eq + 2 AE ⊐ Aq + Eq − 2
and 17. 10. AE. *h* (Q. A − E.)
h *cor.* 7. 2.

PROP.

PROP. XXVII.

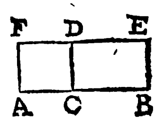


A medial rectangle AB *exceedeth not a medial rectangle* AC *by a rational rectangle* DB.

Upon EF $\dot\rho$, *a* make EG = AB, *a* and EH = AC. The rectangles AB, AC, *i. e.* EG, EH, *b* are $\mu\alpha$; *c* therefore FG and FH are $\dot\rho$ $\top\!\!\sqcup$ EF. Whence, if KG, *d i. e.* DB be $\dot\rho\nu$, *e* then shall HG be $\top\!\!\sqcup$ HK; *f* wherefore HG $\top\!\!\sqcup$ FH. *g* and consequently FGq $\top\!\!\sqcup$ FHq. But FH is $\dot\rho$. *h* therefore is FG $\dot\rho$. but FG was $\dot\rho$ before. *Which is contradictory.*

a *cor.* 16. 6.
b *hyp.*
c 23. 10.
d 3. *ax.* 1.
e 21. 10.
f 13. 10.
g *lem.* 26. 10.
h *sch.* 12. 10.

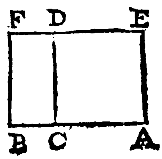
Schol.



1. *A rational rectangle* AE *exceeds a rational rectangle* AD *by a rational rectangle* CE.

For AE *a* $\top\!\!\sqcup$ AD, *b* therefore AE $\top\!\!\sqcup$ CE *c* wherefore CE is $\dot\rho\nu$. *Which was to be dem.*

a *hyp.*
b *cor.* 16. 10.
c *sch.* 12 10

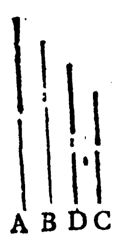


2. *A rational rectangle* AD *joined with a rational rectangle* CF *makes a rational rectangle* AF.

For AD *a* $\top\!\!\sqcup$ CF, *b* wherefore AF $\top\!\!\sqcup$ AD and CF; *c* and so AF is $\dot\rho\nu$. *Which was to be dem.*

a *sch.* 12. 10
b 16. 10.
c *sch.* 12 10

PROP. XXVIII.



To find out medial lines (C *and* D.) *which contain a rational rectangle* CD.

a Take A and B $\dot\rho$ $\top\!\!\sqcup$. *b* make A C :: C. B. *c* and A. B :: C. D. I say the thing required is done. For AB (Cq) *d* is $\mu\nu$, *d* whence C is μ. but because A B :: C. D. *f* therefore C $\top\!\!\sqcup$ D. *g* and consequently D is μ. Moreover by permutation A. C :: B. D. *i. e.* C. B :: B. D. *h* therefore Bq = CD. But Bq is $\dot\rho\nu$. *h* therefore CD is $\dot\rho\nu$. *Which was to be done.*

a *lem.* 21. 10
b 13. 6.
c 12. 6.
d 22. 10.
e *constr.*
f 10. 10.
g 24. 10
h 17. 6.
h *sch.* 12. 10.

In

In numbers, let A be $\surd$ 2; and B $\surd$ 6. therefore C is $v\surd$ 12. make $\surd$ 2. $\surd$ 6 :: $v\surd$ 12. D. or $v\surd$ 4. $v\surd$ 36 :: $v\surd$ 12.. D. then shall D be $v\surd$ 108. but $v\surd$ 12x $v\surd$ 108 = $v\surd$ 1296 = $\surd$ 36 = 6. therefore CD is 6; likewise C. D :: 1. $\surd$ 3. wherefore C $\top\!\top$ D.

PROP. XXIX.

To find out medial right lines commensurable in power only, D *and* E, *containing a medial rectangle* DE.

a Take A, B, C *b* $\top\!\top$. make A D *b* :: D. B. *c* and B. C :: D. E. I say the thing desired is performed.

For AB *d* = Dq. and AB *e* is μr, therefore D is μ; and B *f* $\top\!\top$ C, *g* whence D $\top\!\top$ E. therefore *h* E is μ. Moreover B. C *f* :: D. E. and by permutation B. D :: C. E. *i.e.* D. A :: C. E. *l* therefore DE = AC. But AC *m* is μr. therefore DE is μr. *Which was to be done.*

a *lem.* 21. 10
b 13. 6.
c 12. 6.
d 17. 6.
e 22. 10.
f *constr.*
g 10. 10.
h 24. 10.
k *constr.* *and cor* 4. 5
l 16. 6.
m 22. 6.

In numbers, let A be 20. and B, $\surd$ 200, and C. $\surd$ 80. Therefore D is $\surd\surd$ 80000; and E $v\surd$ 12800. Therefore DE = $\surd\surd$ 1024000000 = $\surd$ 32000. and D. E :: $\surd$. 10. 2. wherefore D $\top\!\top$ E.

Schol.

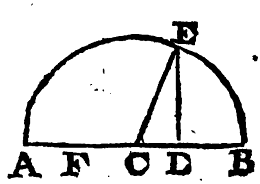
To find out two plane numbers, like or unlike.

Take any four numbers proportional A. B :: C. D it is manifest that AB and CD are like plane numbers. And you may find out as many unlike plane numbers, as you please, by help of *Schol.* 27. 8.

A, 6.	C, 12
B, 4.	D, 8.
AB, 24.	CD, 96.

A, 6.	C 5.
B, 4.	D, 8.
AB, 24.	CD, 40.

Lemma.



To

To find out two square numbers (DEq *and* CDq) *so that the number composed of them* (CEq) *be square also.*

Take AD, DB like plane numbers (of which let both be even, or both odd) *viz.* AD, 24. and DB, 6. The total of these (AB) is 30; the difference (FD) 18. half of whith (CD) is 9. *a* Now the like plane numbers AD, DB, have one mean number proportional, namely DE. therefore it is evident that every of those numbers CE, CD, DE, are rational, and by conseqence CEq (*b* CDq + DEq) is the square number required. a 18. 8. b 47. 1.

Whereby it will be easy to find out two square numbers, the excess of which is a square or not a square number, namely by the same construction *c* shall CEq — CDq be = DEq. c 3. *ax.* 1.

But if AD, DB be plane numbers unlike, the mean proportional line (DE) shall not be a rational number, and so neither shall the excess (DEq) of the square numbers, CEq, CDq. be a square number.

Lemma 2.

2 *To find out two such square numbers* B, C, *as the number compounded of them* D *is not square. Also to divide a square number* A *into two numbers* B, C, *not squares.*

A, 3. B, 9. C, 36. D, 45.

1. Take any square number B, and let C be = 4 B, and D = B + C. I say the thing is done

For B is Q. by the constr. likewise because B. C :: 1. 4 :: Q. Q. *a* therefore C also shall be a square number. But because B + C (D). C :: 5. 4 :: not Q. Q. *b* therefore shall not D be a square number. *Which was to be done.* a 24. 8. b *cor.* 24. 8.

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be some square number. Take D, E, F, plane numbers unlike, and let D be = E + F make D. E :: A. B. and D. F :: A. C I say the thing required is done.

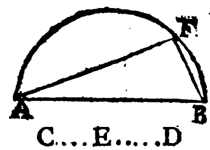
For because D. E + F :: A. B + C, and D = E + F, *a* therefore shall A = B + C. Now suppose B to be square, *b* then A and B, *c* and consequently D and E are like plane numbers. *Which is contrary to the Hyp.* a 14. 5. b 21. *def.* 7 c 26. 8.

The same absurdity will follow if C be supposed a square number, Therefore, &c.

PROP.

PROP. XXX.

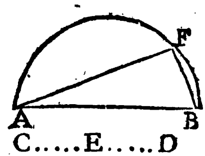
To find out two rational right lines AB, AF, *commensurable only in power, so that the greater* AB *shall be in power more than the less* AF *by the square of a right line* BF *commensurable to it self in length.*



a 1.*lem* 29. 10. Let AB be $\dot{\rho}$. *a* Take the square numbers CD, CE. so that CD — CE (ED) be not Q. *b* and make
b 3.*lem*.10. 10. CD. ED :: ABq. AFq. In a circle described upon the diameter AB *c* fit AF, and draw BF. Then I say
c 1. 4. AB, AF, are the lines required.
d *constr.* For ABq, AFq *d* :: CD. ED. *e* therefore ABq ⊓⊔ AFq,
e 6. 10. but AB is $\dot{\rho}$. *f* therefore AF is also $\dot{\rho}$. But because CD is
f *sch.* 12. 10. Q: and ED not Q: *g* therefore shall AB be ⊓⊔ AF. Moreover by reason of the *h* right angle AFB, is ABq *k*
g 9. 10. = AFq + BFq; therefore seeing ABq AFq :: CD. ED.
h 31. 3. by conversion of proportion shall ABq. BFq :: CD. CE ::
k 47. 1. Q. Q. *l* therefore AB ⊓⊔ BF. *Which was to be done.*
l 9. 10. In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9 5 :: 36. (Q: 6.) AFq. then AFq shall be 20. and consequently AF √. 20. therefore BFq = 36 — 20 = 16. wherefore BF is 4.

PROP. XXXI.

To find out two rational lines AB, AF *commensurable only in power, so that the greater* AB *shall be in power more than the less* AF *by the square of a right line* BF *incommensurable to it self in length.*



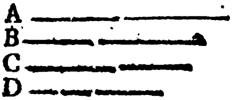
a 1.*lem*.29. 10. Let AB be $\dot{\rho}$. *a* Take the square numbers CE, ED, so that CD = CE + ED be not Q. and in the rest follow the construction of the preced. *prop.* I say then the thing required is done.

For, as above, AB, AF, are $\dot{\rho}$ ⊓⊔. also ABq. BFq ::
b 9. 10. CD. ED. therefore since CD is not Q. AB, BF *b* shall be ⊓⊔. *Which was to be done.*

In

In numbers, let there be AB, 5. CD, 45. CE = 36 ED = 9. Make 45. 9 :: 25 (ABq.) 5 (AFq) therefore AF = √5. consequently BFq = 45 − 25 = 20. wherefore BF = √20.

PROP. XXXII.



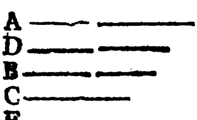
To find out two medial lines C, D, *commensurable only in power, comprehending a rational rectangle* CD, *so that the greater* C *be more in power than the lesser* D *by the square of a right line commensurable in length to the greater.*

a Take A and B ρ̇ ⊓; so as √ Aq − Bq ⊔ A *b* and make A. C :: C. B. *c* and A. B :: C. D. I say the thing is done. — a 30. 10. b 13. 6. c 12. 6.

For because A and *d* B are ρ̇ ⊓. *e* therefore shall C (*f* √ AB) be μ. *g* and thence also C ⊓ D *h* therefore D is likewise μ. Furthermore, whereas A. B *d* :: C. D; and by permutation A. C :: B. D :: C. B; and Bq is ρ̇ν. therefore shall CD (*k* Bq) be ρ̇ν. Lastly, because √ Aq − Bq *d* ⊔ A, *l* shall √ Cq − Dq be ⊔ C. therefore, &c. But if √ Aq − Bq ⊔ Aq, then shall √ Cq − Dq be ⊔. C — d *constr.* e 22. 10. f 17. 6. g 10. 10. h 24. 10. k 17. 6. l 15. 10.

In numbers; let there be A 8, B √ 48 (√: 64 − 16) therefore C = √ AB = v √ 3072. and D = v √ 1728. wherefore CD = v √ 5308416 = √ 2304.

PROP. XXXIII.



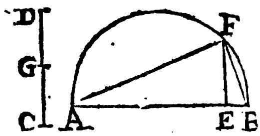
To find out two medial lines D, E, *commensurable in power only, comprehending a medial rectangle* DE, *so that the greater* D *shall be more in power than the less* E, *by the square of a right line commensurable to the greater length.*

a Take A and C ρ̇ ⊓, so that √ Aq − Cq ⊔ A, *b* take also B ⊓ A and C, and make A. D *c* :: D. B *d* :: C. E then D, and E are the lines sought for. — a 30. 10. b *lem.* 21. 10.

For because A and C *e* are ρ̇, *e* and B ⊓ A and C, *f* therefore shall B be ρ̇, and D (*g* √ AB) *g* shall be μ. But because A. D :: C. E, therefore by permutation A. C :: D. E wherefore seeing A ⊓ C, therefore D shall be ⊓ E. therefore E is μ. Furthermore, *l* because D. B :: C. E. and BC is μν. also DE, equal to it, is μν. Lastly, because A. C :: D. E *e* seeing √ Aq − Cq ⊔ A. therefore √ Dq − Eq ⊔ D. therefore, &c. But if √ Aq − Cq ⊔ A then √ Dq − Eq ⊔ Eq. — c 13. 6. d 12. 6. e *constr.* f *sch.* 12. 10. g 22. 10. h ro. 10. k 24. 10. l 22. 10. m 16. 6. n 15. 5.

In numbers, let there be A 8, C $\surd$ 48. B $\surd$ 28. then D ϖ $\surd$ 3072. and E ϖ $\surd$ 588. wherefore D. E :: 2. $\surd$ 3. and DE $=$ $\surd$ 1344.

PROP. XXXIV.



To find out two right lines AF, BF, *incommensurable in power, whose squares added together make a rational figure, and the rectangle contained under them medial.*

a 31. 10. *a* Let there be found AB,
b 10. 1. CD, ρ ⊤⊥ ; so that $\surd$ ABq — CDq ⊤⊥ AB, divide CD
c 28 6. equally in G. *c* make the rectangle AEB $=$ GCq. Up-
d 12. 6. on AB the diameter draw a semicircle AFB, erect the
e *cor.* 8 6. perpendicular EF, and draw AF, BF. These are the
& 17. 6. lines required.

f 7. 5. For AE. BE *d* :: BA x AE. AB x BE. But BA x AE
g 19. 10. *e* $=$ AFq; and AB x BE $=$ FBq. *f* therefore AE. EB
h 10. 10. :: AFq. FBq. therefore since AE *g* ⊤⊥ EB, *h* AFq shall
k 31. 3 & be ⊤⊥ FBq. Moreover ABq (*k* AFq $+$ FBq) *l* is $\rho\nu$. Last-
47. 1. ly EFq *l* $=$ AEB *l* $=$ CGq. *m* therefore EF $=$ CG.
l *constr.* therefore CD x AB $=$ 2 EF x AB. But CD x AB *n* is $\mu\nu$.
m 1 *ax.* 1. *o* therefore AB x EF, *p* or AF x FB is $\mu\nu$. *Which was to*
n 22. 10. *be demonstrated.*
o 24. 10.
p *sch.* 22. 6.

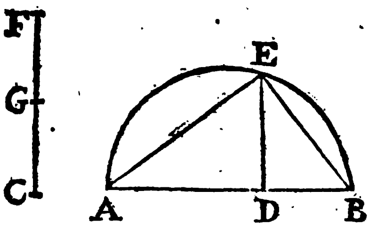
The Explication of the same by numbers.

Let AB be 6. CD $\surd$ 12, then CG $= \surd \frac{12}{4} = \surd$ 3. But AE $=$ 3 $+$ $\surd$ 6. and EB $=$ 3 $-$ $\surd$ 6. whence AF shall be $\surd$: 18 $+$ $\surd$ 216. and FB $\surd$: 18 $-$ $\surd$ 216. Also AFq $+$ FBq is 36, and AF x FB $=$ $\surd$ 108.

But AE is found in this manner. Because BA (6.) AF :: AF. AE. therefore 6 AE $=$ AFq $=$ AEq $+$ 3 (EFq) therefore 6 AE $-$ AEq $=$ 3. Put 3 $+$ e $=$ AE. then 18 $+$ 6e $-$ 9. $-$ 6e $-$ ee, that is, 9 $-$ ee $=$ 3. or ee $=$ 6 wherefore e $=$ $\surd$ 6. and so AE $=$ 3 $+$ $\surd$ 6.

PROP

PROP. XXXV.

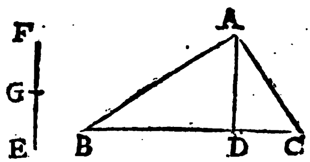


To find out two right lines AE, EB, *incommensurable in power, whose squares added together make a medial figure, and the rectangle contained under them rational.*

a Take AB and CF μ ⊓, so that AB x CF be ῥν, and √ ABq — CFq ⊔ AB, and let the rest be done as in the prec. *prop.* AE, EB are the lines required. a 32. 10.

For, as it is shewn there, AEq ⊔ EBq. also ABq (AEq + EBq) μν. and lastly AB x CF *b* is ῥν. *c* therefore also AB x DE, that is, AE x EB, is ῥν. therefore, &c. b *constr.* c *sch.* 12. 10. d *sch.* 22. 6.

PROP. XXXVI.

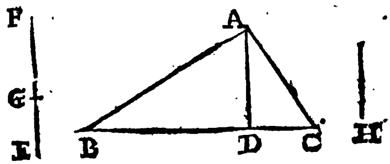


To find out two right lines BA, AC, *incommensurable in power, whose squares added together make a medial figure, and the rectangle also contained under them medial, and incommensurable to the figure composed of the squares.*

a Take BC and EF μ ⊓, so that BC x EF be μν. and √ BCq — EFq ⊔ BC, and so forward, as in the prec. BA, AC, shall be the lines sought for. a 33. 10

For (as above) BAq ⊔ ACq, also BAq + ACq is μν. and BA x AC is μν. Lastly, BC *b* ⊔ EF, and *c* so BC ⊔ EG; likewise BC. EG *d* :: BCq. BC x EG (BC x AD, or BA x AC) *e* therefore BCq (ABq + ACq) ⊔ BA x AC. therefore, &c. b *constr.* c 13. 10. d 1. 6. e 14. 10.

Schol.



To find out two medial lines incommensurable both in length and power.

a 36. 10. *a* Take BC μ, and let BA x AC be $\mu\nu$, and $\sqcup$ BCq
b 13. 6. (BAq + ACq) *b* make BA. H :: H. AC. then I say BC and
c 17. 6. H are μ $\not\sqcup$. For BC is μ. *a* and BA x AC (*c* Hq) is $\mu\nu$.
d 14. 10. wherefore H is also μ. *d* Likewise BA x AC $\not\sqcup$ BCq; therefore Hq $\sqcup$ BCq. therefore, &c.

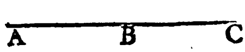
Here begin the senaries of lines irrational by composition.

PROP. XXXVII.

If two rational lines AB, BC, commensurable only in power, are added together, the whole line AC is irrational, and is called a binomial line, or of two names.

a hyp. For because AB *a* $\sqcup$ BC, thence *b* shall ACq be $\sqcup$
b lem. 26. 10. ABq. But AB *a* is $\dot{\rho}$. *c* therefore AC is $\dot{\rho}$. *Which was to be demonstrated.*
c 11. def. 10.

PROP. XXXVIII.

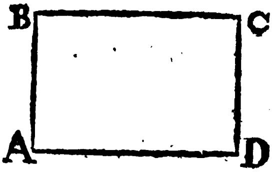


If two medial lines AB, BC, commensurable in power only, are compounded, and contain a rational rectangle, the whole line AC is irrational, and called a first bimedial line.

a hyp. For because AB *a* $\sqcup$ BC, *b* shall ACq be $\sqcup$ AB x
b lem. 26. 10. BC, $\dot{\rho}\nu$. *c* therefore AC is $\dot{\rho}$. *Which was to be dem.*
c 11. def. 10.

Lemma.

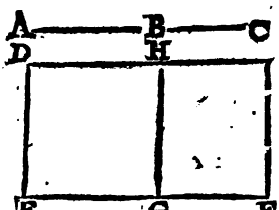
Lemma.



A rectangle AC, contained under a rational line AB and an irrational line BC, is irrational.

For if the rectangle AC be affirmed ῥν, *a* then be- a *hyp.*
cause AB is ῥ, *b* the breadth b 21. 10.
BC shall be also ῥ. *against the Hyp.*

PROP. XXXIX.



If two medial lines AB, BC, commensurable only in power, containing a medial rectangle, are compounded, the whole line AC shall be irrational, and is called a second bimedial line.

Upon the propounded
line DE ῥ *a* make the rectangle DF = ACq; *b* and DG a *cor.* 16. 6.
= ABq + BCq. b 47. 1. &
Because ABq *c* ⊓⊔ BCq, *d* therefore ABq + BCq, *i. e.* 11. 6,
DG, ⊓⊔ ABq: but ABq *e* is μν, *e* therefore DG is μν. But c *hyp.*
the rectangle ABC is taken μν, *e* and consequently d 16. 10.
2 ABC (*f* HF) is μν. *g* therefore EG and GF are ῥ. e 24. 10.
Also because DG *h* ⊓⊔ HF; and DG. HF :: *k* EG. f 4. 2.
GF; *l* therefore EG ⊓⊔ GF. *m* therefore the whole EF g 23. 10.
is ῥ. *n* wherefore the rectangle DF is ῥν. *o* therefore √ h *lem.* 26.
DF, *i. e.* AC, is ῥ. *Which was to be demonstrated.* 10.
k 1. 6.
l 10. 10.
m 37. 10.

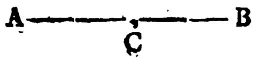
PROP. XL.

If two right lines AB, BC, commensurable only in power, are added together, making that which is composed of their squares rational, and the rectangle contained under them medial, the whole right line AC is irrational, and is called a Major line.

n *lem.* 38. 10.
o 11. *def.* 10.

a *hyp.* For whereas ABq + BCq *a* is ρ̇ν, and *b* ⊔⊔ 2 ABC *c*
b *sch.* 12. μν; and so ACq (*d* ABq + BCq + 2 ABC) *e* ⊔⊔ ABq
10. + BCq ρ̇ν. *f* therefore shall AC be ρ̇. *Which was to be*
c *hyp. and* *demonstrated*
24. 10.
d 4. 2.
e 17. 10.
f 11. *def.* 10.

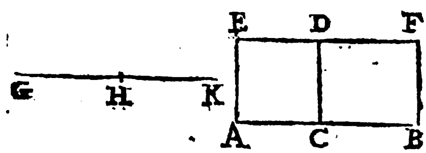
PROP. XLI.



If two right lines AC, CB, *incommensurable in power, are added together, having that which is made of their squares added together medial, and the rectangle contained under them rational, the whole right line* AB *shall be irrational, and is called A line containing in power a rational and a medial rectangle.*

a *hyp. and* For 2 rectangles ACB *a* ρ̇ν, *b* ⊔⊔ ACq + CBq *c* μν *d*
sch. 12. 10. therefore 2 ACB *d* ⊔⊔ ABq. wherefore *e* AB is ρ̇. *Which*
b *sch.* 12. *was to be demonstrated.*
10.
c *hyp.*
d 17 10.
e 11. *def.*
10.

PROP. XLII.

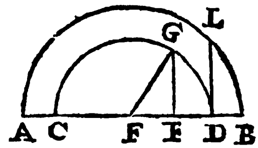


If two right lines GH, HK, *incommensurable in power, are added together, having both that which is composed of their squares medial, and the rectangle contained under them medial, and incommensurable to that which is composed of their squares, the whole right line* GK *is irrational, and is called A line containing in power two medial figures.*

Upon the propounded line FB ρ̇ make the rectangles AF = GKq, and CF = GHq + HKq. Because GHq + HKq (CF) *a* is μν, the breadth CB *b* shall be ρ̇. Also be-
a *hyp.* cause 2 rectangles GHK (*c* AD) *a* is μν, therefore AC *b*
b 23. 10. shall be ρ̇. Moreover because the rectangle AD *a* ⊔⊔. CF,
c 4. 2. *d* and AD. CF :: AC. CB, *e* thence shall AC be ⊔⊔ CB.
d 2. 6. *f* wherefore A is *g* ρ̇. therefore the rectangle AF. *i. e.*
e 10. 10. GKq is ρ̇ν; *h* and consequently GK is ρ̇. *Which was to*
f 37. 10. *be demonstrated.*
g *lem.* 38.
10.
h 11. *def.*
10.

PROP.

PROP. XLIII.



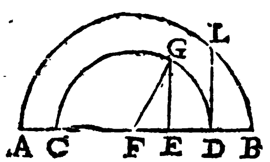
A line of two names, or binominal, AB, *can at one point only* D *be divided into its names,* AD, DB.

If it be possible, let the binominal line AB be divided at the point E, into other names AE, EB. It is manifest that the line AB is in both cases divided unequally, since AD $\sqsupset$ DB, and AE $\sqsupset$ EB.

Because the rectangles ADB, AEB *a* are μα; *a* and each of ADq, DBq, AEq, EBq is ρα. *b* and so ADq + DBq *b* and AEq + EBq are also ρα. *b* therefore ADq + DBq $-:$ AEq + EBq *c i. e.* 2 AEB − 2 ADB is ρν. *d* therefore AEB − ADB is ρν. therefore μν exceeds μν by ρν. *e Which is absurd.*

a 37. 10.
b *sch.* 27. 10.
c *sch.* 5. 2.
d *sch.* 12. 10.
e 27. 10.

PROP. XLIV.

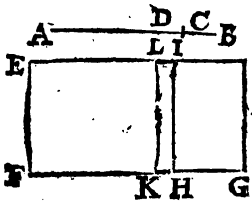


A first bimedial line AB *is in one point only* D *divided into its names* AD, DB.

Conceive AB to be divided into other names AE, EB; whereupon every one ADq, DBq, EBq, will be *a* μα. and the rectangles ADB, AEB, and the doubles of them, ρα. *b* therefore 2 AEB − 2 ADB. *c i. e.* ADq + DBq $-:$ AEq + EBq is ρν. *d Which is absurd.*

a 38. 10.
b *sch.* 27. 10.
c *sch.* 5. 2.
d 27. 10.

PROP. XLV.

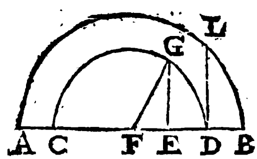


A second bimedial line AB, *is divided into its names* AC, CB, *only at one point* C.

Suppose there were other names AD, DB. Upon the propounded line EF $\dot{\rho}$ make the rectangles EG = ABq, and EH = ACq + CBq, as also EK = ADq + DBq.

a 39. 10. Because ACq, BCq *a* are $\mu\alpha$ ⊏⊐; *b* ACq + CBq (EH)
b 16 *and* shall be $\mu\nu$. *c* therefore the breadth FH is $\dot{\rho}$. *a* moreover
24. 10. the rectangle ACB, *d* and so 2 ACB (*e* IG) is $\mu\nu$. *c* there-
c 29. 10. fore HG is also $\dot{\rho}$. And since EH is *f* ⊏⊐ IG, *g* and EH.
d 24. 10 IG :: FH. HG. *h* therefore FH, HG shall be ⊏⊐. *k* there-
e 4. 2. fore FG is a binomial, whose names are FH, HG. By
f *lem.* 26. the same reason FG is binomial, and the names of it FK,
10. KG: *contrary to the* 43. *of this Book.*
g 1. 6.
h 10. 10.
k 37. 10.

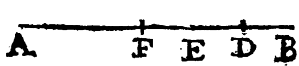
PROP. XLVI.



A Major line AB *is at one point only* D *divided into its names* AD, DB.

Imagine other Names AE, EB whereupon the rectan-
a 40. 10. gles ADB, AEB, *a* $\mu\alpha$. *a* and as well ADq + DBq, as
b *sch.* 27. AEq + EBq are $\dot{\rho}\nu$. *b* therefore ADq + DBq −: AEq
10. + EBq, *c i. e.* 2 AEB — 2 ADB is $\dot{\rho}\nu$. *d Which is impos-*
c *sch.* 5. 2. *sible.*
d 17. 10.

PROP. XLVII.



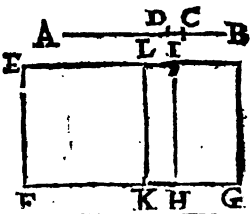
A line AB *containing in power a rational and a medial figure is divided at one point only* D *into its names* AD, DB.

Con-

Conceive other names AE, EB. then both AEq + EBq, and ADq + DBq are *μα*. *a* and the rectangles a 41. 10.
AEB, ADB are ρ́*α*. *b* therefore 2 AEB — 2 ADB, *c i. e.* b *sch.* 27.
ADq + DBq AEq + EBq is ρ́*ν*. *d Which is absurd.* 10.
c *sch.* 5. 2.

PROP. XLVIII. d 27. 10.

A line AB *containing in power two medial rectangles, is at one point only* C *divided into its names* AC, CB.



If you would have AB to be divided into other names AD, DB, draw upon the line compounded EF ρ́ the rectangles EG = ABq, and EH = ACq + CBq, and EK = ADq + DBq. then because ACq + CBq, namely EH, *a* a 42. 10.
is *μν*, *b* the breadth FH shall be ρ́. Also because 2 ACB, b 23. 10.
c that is, IG, is *a μν*, HG *b* shall be likewise ρ́. There- c 4. 2.
fore, whereas EH *a* ⊔ IG, and EH. IG *d*:: FH, HG, d 1. 6.
thence FH *e* shall be ⊔ HG, *f* therefore FG is a bino- e 10. 10.
mial, and the names of it FH, HG. In like manner FK, f 37. 10.
KG shall be the names of it, *against the* 43. *of this Book.*

Second Definitions.

A Rational line being propounded, and the binomial divided into its names, the greater of whose names is more in power than the less by the square of a right line commensurable to the greater in length; then

I. If the greater name be commensurable in length to the rational line propounded, the whole line is called a binomial line.

II. But if the lesser name be commensurable in length to the rational line propounded, the whole line is called a second binomial.

III. If neither of the names be commensurable in length to the rational line propounded, it is called a third binomial.

Furthermore, if the greater name be more in power than the less, by the square of a right line incommensurable to the greater in length, then

IV. If the greater name be commensurable to the propounded rational line in length, it is called a fourth binomial.

V. If the lesser name be so, a fifth.

VI. If neither, a sixth.

PROP. XLIX.

A....4 C.:....5 B

D———— ————

E———F-———G

H——-

To find out a first binomial line, EG.

a sch. 29. 10. b 2 lem. 10. 10. *a* Take AB, AC, square numbers, whose excess CB is not Q. let D be propounded ῥ. *b* Take EF ⊓⊔ D, and *c* make AB. CB :: EFq, FGq. then EG shall be a 1 bin. (c 3 lem. 10. 10.)

d constr. For EF *d* ⊓⊔ D. *e* therefore EF is ῥ. *f* also EFq ⊓⊔ FGq. *g* therefore FG is also ῥ. likewise *d* because EFq. FGq :: AB. CB :: Q. not Q. *h* therefore EF ⊓⊔ FG. Lastly, because by conversion of proportion, EFq. EFq − FGq :: AB. AC :: Q. Q. thence EF *k* shall be ⊓⊔ √ EFq − FGq. *l* therefore EG is a 1 binominal. *Which was to be done.* (e 6. def. 10. f 6. 10. g sch. 12. 10. h 9. 10. k 9. 10.)

l 1. def. 48. 10. In numbers thus; let there be D 8. EF 6. AB 9. CB 5 wherefore because 9. 5 :: 36. 20. therefore FG is √ 20. and consequently EG is 6 + √ 20.

PROP. L.

A....4 C.....5 B

D————————

E———F-———G

H————

To find out a second binomial line, EG.

Take AB and AC square numbers, the excess of which is CB not Q. Let the line D be propounded ῥ. take FG ⊓⊔ D, and make CB. AB :: FGq. EFq. then EG will be the line desired.

Prove it as the prec.

For FG ⊓⊔ D. wherefore FG is ῥ. Also EFq ⊓⊔ FGq therefore EF is ῥ. Likewise because FGq. EFq :: CB. AB :: not Q. Q. thence FG is ⊓⊔ EF. Lastly, seeing CB. AB :: FGq. EFq. and inversly AB. CB :: EFq. FGq. therefore as in the foregoing Prop. EF ⊓⊔ √ EFq − FGq. *a* whereby EG is a 2 binomial. *Which was to be done.*

a 2. def. 48. 10. In numbers; let there be D 8, FG 10, AB 9, CB 5. then EF is √ 180, wherefore EG is 10 + √ 180.

PROP. LI.

A....4 C.....5 B

L......6

G————

D——————F———F

H——————

To find out a third binomial line, DF.

a sch. 19. 10. *a* Take AB, AC, square numbers, the excess of which CB is not Q and let L be a number not Q next greater than CB, *viz.* by a unit or two. Let G be the line propounded

pounded $\dot{\rho}$. *b* Make L. AB :: Gq. DEq. *b* and AB. CB :: DEq. EFq. then DF shall be a third binomial. b 3*lem*.10. 10.

For because DEq *c* ⊓⊔ Gq, *d* DE is $\dot{\rho}$. also Gq. DEq :: L. AB :: not Q Q. *e* therefore G '⊓⊔ DE. Likewise since DEq *e* ⊓⊔ EFq, *d* also EF is $\dot{\rho}$. Moreover because DEq. EFq :: AB. CB :: Q. not Q. *f* is DE '⊓⊔ EF. and since by constr. and equality Gq. EFq :: L. CB :: not Q. Q. (for *g* L and CB are not like plane numbers) *h* therefore shall G be also '⊓⊔ EF. Lastly, as in the prec. prop. $\sqrt{}$ DEq — EFq ⊓⊔ DE. *k* therefore DF is a 3 binomial. *Which was to be done.* c *constr*. 6 10. d *sch*. 12. 10. e 6. 10. f 9. 10. g *sch* 27 8. h 9. 10. k 3 *def*. 48. 10.

In numbers; let there be AB, 9. CB, 5. L, 6. G, 8, then shall be DE $\sqrt{96}$, and EF $\sqrt{4\frac{8}{9}0}$. wherefore DF $= \sqrt{96} + \sqrt{4\frac{8}{9}0}$.

PROP. LII.

A ... 3 C ..... 6 B
G ———— ——
D ———— E ———— F
H ————

To find out a fourth binomial line DF.

a Take any square number AB, and divide it into AC, CB not squares. Let G be the line propounded $\dot{\rho}$. *b* take DE ⊓⊔ G, *c* and make AB. CB :: DEq. EFq, then DF shall be a 4 binomial. a *sch*. 29. 10. b 2 *lem*. 10. 10.

For, as in the 49 of this Book, DF may be shewn to be a binomial, and also because by constr. and conversion of proportion DEq. DEq — EFq :: AB. AC :: Q. not Q. *d* shall DE be '⊓⊔ $\sqrt{}$ DEq — EFq. *e* therefore DF is a 4 binomial. c 3. *lem*. 10. 10. d 9 10. e 4. *def*. 48. 10.

In numbers, let G be 8, DE, 6. then EF shall be $\sqrt{24}$. therefore DF is $6 + \sqrt{24}$.

PROP. LIII.

A ... 3 C ...... 6 F
G ————
D ———— E ———— F
H ————

To find out a fifth binomial line DF.

Take any square number AB, whose segments AC, CB are not Q. Let G be the line propounded $\dot{\rho}$. take EF ⊓⊔ G. and make CB AB :: EFq. DEq. then shall DF be a 5 binomial.

For DF shall be a binomial as in the 50. of this Book, and because by construction, and inversion, DEq. EFq :: AB. CB and so by conversion of proportion, DEq. DEq — EFq :: AB, AC :: Q. not Q. *a* therefore shall DE be a 9 10.

'⊓⊔

b 5. *def* 48. 10. ⊔ $\surd$ DEq − EFq. *b* therefore DF is a 5 binomial. *Which was to be done.*

In numbers, let there be G, 7. EF, 6. then DE shall be $\surd$ 54. wherefore DF is 6 + $\surd$ 54.

PROP. LIV.

A.....5 C........7 B
L.........9
G——— - ———
D —— E ——F
H ———

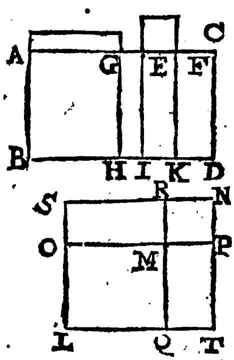
To find out a sixth binomial line

Take AC, CB, prime numbers, so that AC + CB (AB) be not Q. take also any number square L. Let G be the line propounded
a 3. *lem* 10. 10. *p.* *a* and make L. AB :: Gq DEq, and AB. CB :: DEq. EFq. then DF shall be a 6 binomial.

For DF may be demonstrated binomial as in the 51. of this Book; and also by reason that DE and EF ⊔ G. lastly likewise because by constr. and conversion of proportion DEq. DEq − EFq :: AB. AC :: not Q.Q. (For AB
b *sch* 27. 8. is prime to AC, *b* and so unlike to it) *c* therefore DE ⊔
c 9. 10 $\surd$ DEq − EFq. *d* therefore DF is a 6 binomial. *Which*
d 6 *def* 48. 10. *was required.*

In numbers, let there be G 6. DE $\surd$ 48. then EF shall be $\surd$ 28. wherefore DF is $\surd$ 48 + $\surd$ 28.

Lemma.



Let AD *be a rectangle, and the side thereof* AC *divided unequally in* E; *also let the lesser portion* EC *be equally divided in* F.
a 28. 6. *upon the line* AE a *make the rectangle* AGE = EFq, *and from*
b 31. 1. *the points* G, E, F b *draw* GH, EI, FK, *parallel to* AB, c *Let*
c 14. 2. *the square* LM *be made equal to the rectangle* AI, *and upon* OMP *produced the square* MN = GI, *and let the right lines* LOS, LQT, NRS, NPT *be produced.*

I say 1. MS, MT, are rectangles. For by reason of the
a *sch.* 15. 1. right angles of the squares OMQ, RMP, *a* shall QMR
b 13. 1. be a right line. *b* therefore RMO, QMP, are right angles, wherefore the parallelograms MS, MT, are rectangles.

2. *Hence*

2. *Hence it is plain that* LS *c* = LT, *and consequently that* LN *is a square.* c 2. *ax.* 1.

3. *The rectangles* SM, MT, EK, FD *are equal.* For because the rectangle AGE *d* = EFq. *e* thence shall AG. EF :: EF. GE, *f* and so AH. EK :: EK. GI. that is by constr. LM. EK :: EK. MN. *g* but LM. SM :: SM. MN. therefore EK *h* = SM *k* = FD *l* = MT. d *hyp.* e 17. 6. f 1. 6. g *sch* 22. 6.

4. *Hence* LN *m* = AD. h 9. 5.

5. *Because* EC *is equally divided in* F, *n it is plain that* EF, FC, EC *are* ⊓⊔. k 36. 1. l 43. 1.

6. If AE ⊓⊔ EC, and AE ⊓⊔ $\surd$ AEq − ECq, *o* then shall AG, GE, AE, be ⊓⊔. also, because AG. GE :: AH. GI, *p* therefore shall AH, GI, *i. e.* LM, MN, be ⊓⊔. Likewise thereupon, m 2. *ax* 1. n 16. 10. o 18. *and* 16. 10.

7. OM ⊓⊔ MP. For by the Hyp AE ⊓⊔ EC *q* therefore EC ⊓⊔ GE. *q* wherefore EF ⊓⊔ GE. but EF. GE :: EK. GI. *r* therefore EK ⊓⊔ GI that is, SM ⊓⊔ MN. but SM. MN :: OM. MP. *r* therefore OM ⊓⊔ MP. p 10. 10. q 14. 10. r 10. 10.

8. If AE be supposed ⊓⊔ $\surd$ AEq − ECq, it is apparent that AG, GE, AE, are ⊓⊔. whence LM ⊓⊔ MN. for AG. GE :: AH. GI :: LM. MN. f 19. *and* 17. 10.

These being well considered, we shall easily dispatch the six following Propositions.

PROP. LV.

If a space AD *be contained under a rational line* AB, *and a first binomial line* AC (AE + EC) *the right line* OP *which containeth that space in power is irrational, and called a binomial line.*

All that being supposed which is described and demonstrated in the next foregoing *Lemma*, it is manifest that the right line OP containeth in power the space AD. *a* Likewise AG, GE, AE, are ⊓⊔. therefore seeing AE, *b* is ρ ⊓⊔ AB. *c* shall also AG and GE be ρ ⊓⊔ AB. *d* therefore the rectangles AH, GI, that is, the squares LM. MN are $\rho\alpha$. therefore OM, MP are ρ *e* ⊓⊔. *f* and consequently OP is a binomial. *Which was to be demonstrated.* a *hyp and lem* 54. 10. b *hyp.* c *sch.* 12. 10. d 20. 10. e *lem.* 54. 10. f 37. 10.

In numbers, let there be AB 5. AC 4 + $\surd$ 12. wherefore the rectangle AD = 20 + $\surd$ 300 = to the square LN. therefore OP is $\surd$ 15 + $\surd$ 5. namely a 6 binomial.

PROP.

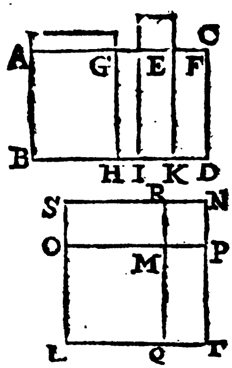
PROP. LVI.

If a space AD be comprehended under a rational line AB, and a second binomial AC (AE + EC) the right line OP, which containeth that space AD in power, is irrational, and called a first medial line.

The foresaid *Lemma* of the 54. of this Book being
a *hyp. and* again supposed, then shall OP be = √ AD. *a* also
lem 54 10. AE, AG, GE, are ⊓⊔. therefore since AE *b* is ῥ ⊓⊔
b *hyp.* AB, likewise AG, GE *c* shall be ῥ ⊓⊔ AB, therefore the rec-
c *sch* 12. 10. tangles AH, GI, *i. e.* OMq, MPq. *d* are μα. *e* Moreover
d 22. 10. OM ⊓⊔ MP. Lastly, EF ⊓⊔ EC, and EC *f* ⊓⊔ AB.
e *lem.* 54. *g* wherefore EK, *i. e.* SM, or OMP, is ῥν. *h* Consequent-
10. ly OP is a first bimedial. *Which was to be dem.*
f *hyp.* 12. In numbers, let there be AB 5, and AC, √ 48 : + 6.
10. then the rectangle AD = √ : 1200 + 30 = OPq. there-
g 20. 10. fore OP is *v* √ 675. + *v* √ 75. *viz.* a first bimedial.
h 38. 10.

See Scheme 57.

PROP. LVII.



If a space AD be contained under a rational line AB, and a third binomial line AC (AE + EC) the right line OP which containeth in power the space AD, is irrational, and called a second bimedial line.

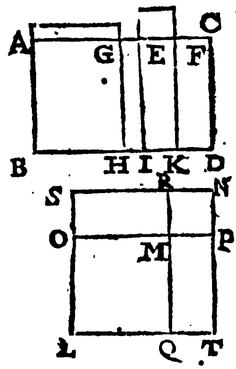
a *hyp. and* As above, OPq = AD. also
22. 10. the rectangles AH, GI, that is,
b 39. 10. OMq, MPq are μα. *a* Likewise FK, or OMP is μν. *b* therefore OP is a second bimedial.

In numbers, let there be AB 5. AC √ 32 + √ 24 wherefore AD is √ 800 + √ 600 = OPq. and so OP is *v* √ 450 + *v* √ 50, that is, a second bimedial.

PROP.

PROP. LVIII.

If a space AD *be comprehended under a rational line* AB *and a fourth binomial* AC (AE + EC) *the right line* OP *containing the space* AD *in power, is that irrational line which is called a Major line.*



For again, OMq *a* ⊏⊐ MPq; and the rectangle AI, *i. e.* OMq + MPq *b* is ῥν. *c* also EK or OMP is μν. *d* therefore OP (√ AD) is a *Major* line. *Which was to be demonstrated.* a *lem.* 54. 10. b *hyp* and 20. 10. c *hyp* and 22. 10. d 40. 10.

In numbers, let there be AB 5. and AC 4 + √ 8. then the rectangle AD is 20 + √ 200. wherefore OP is √ : 20 + √ 200.

PROP. LIX.

If a space AD *be contained under a rational line* AB, *and a fifth binomial* AC, *the right line* OP *which containeth the space* AD *in power, is that irrational line, which is a line containing a rational and a medial rectangle in power.*

Again OMP ⊏⊐ MPq. and the rectangle AI or OMq + MPq is μν. *a* Likewise the rectangle EK or OMP is ῥν. *b* therefore OP (√ AD) contains in power ῥν and μν. *Which was to be dem.* a *as in the prec.* b 41. 10.

In numbers, let there be AB 5. and AC 2 + √8. then the rectangle AD = 10 + √ 200 = OPq. Wherefore OP is √ : 10 + √ 200.

PROP. LX.

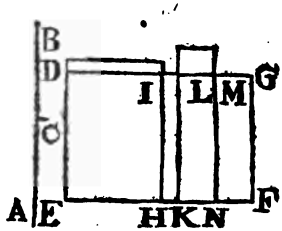
If a space AD *be contained under a rational line* AB *and a sixth binomial* AC (AE + EC) *the line* OP *containing the space* AD *in power is irrational, which containeth in power two medial rectangles.*

As often before, OMq ⊏⊐ MPq, and OMq + MPq is μν. and also the rectangle (EK) OMP is μν. *a* therefore OP = √ AD contains in power 2 μα. *Which was to be dem.* a 42. 10.

In

In numbers, let there be AB 5. AC $\sqrt{12} + \sqrt{8}$ therefore the rectangle AD or OPq is $\sqrt{300} + \sqrt{200}$ and so OP is $\sqrt{:\ \sqrt{300} + \sqrt{200}}$.

Lemma.



Let a right line AB *be unequally divided in* C, *and let* AC *be the greater segment, and upon some line* DE *apply the rectangles* DF = ABq, *and* DH = ACq, *and* IK = CBq, *and let* LG, *be divided equally in* M, *and also* MN *drawn parallel to* GF.

a 4. 2. *and* 3. *ax.* 1. — I say, 1. The rectangle ACB is = LN or MF. *a* For 2 ACB = LF.

b 7. 2. — 2. DL ⊏ LG. for DK (ACq + CBq.) *b* ⊏ LF (2

c 1. 6. — ACB) therefore since DK, LF are of equal altitude, *c* DL shall be ⊏ LG.

d 16 10. — 3. If AC ⊓ CB, *d* then shall the rectangle DK be ⊓ ACq and CBq.

e *lem.* 26. 10. — 4. *Also* DL ⊔ LG. For ACq + CBq *e* ⊔ 2 ACB, *i e* DK ⊔ LF, but DK. LF *e* :: DL. LG, *f* therefore

f 10. 10. — DL ⊔ LG.

5. *Moreover* DL ⊔ $\sqrt{}$ DLq — LGq. For ACq. ACB

g 1. 6. — *g* :: ACB. CBq. that is DH. LN :: LN. IK. *c* wherefore

h 17. 6. — DI. LM :: LM IL, *b* therefore DI x IL = LMq. there-

k *hyp.* — fore seeing ACq *k* ⊔ CBq, that is, DH ⊔ IK, and *l*

l 10 10. — so DI ⊔ IL, *m* shall DL be ⊔ $\sqrt{}$ DLq — LGq *Which*

m 18. 10. — *was to be dem*

n 19. 10. — 6. *But if* ACq *be put* ⊔ CBq, n *then shall* DL be ⊔ $\sqrt{}$ DLq — LGq.

This Lemma is preparatory to the six following Propositions.

PROP.

PROP. LXI.

The square of a binomial line (AC + CB) *applied unto a rational line* DE, *makes the latitude* DG *a first binomial line.*

Those things being supposed, which are described and demonstrated in the next preceding *Lemma*; because AC, CB *a* are ρ́ ⊓⊔, *b* the rectangle DK shall be ⊓⊔ ACq. *c* and so DK is ρ́ν. *d* therefore DL ⊓⊔ DE ρ́. but the rectangle ACB, and so 2 ACB (LF) *e* is μν. *f* therefore the latitude LG is ρ́ '⊓⊔ DE. *g* therefore also DL '⊓⊔ LG also DL ⊓⊔ √ DLq − LGq. from whence *k* it follows that DG is a first binomial. *Which was to be demonstrated.*

a *hyp.*
b *lem.* 60. 10.
c *sch* 12. 10.
d 21. 10.
e 22. *and* 24 10.
f 23. 10.
g 13 10.
h *lem.* 60. 10.
k 1. *def.* 48. 10.

PROP. LXII.

The square of a first bimedial line (AC + CB) *being applied to a rational line* DE, *makes the latitude* DG *a second binomial line.*

The aforesaid *Lemma* being again supposed; The rectangle DK ⊓⊔ ACq. *a* therefore DK is μν. *b* therefore the latitude DL is ρ́ ⊓⊔ DE But because the rectangle ACB, and so LF (2 ACB) *c* is ρ́ν, *d* shall LG be ρ́ '⊓⊔ DE, *e* therefore DL, LG are ⊓⊔. *f* also DL ⊓⊔ √ DLq − LGq. *g* from whence it is clear that DG is a second binomial. *Which was to be dem.*

a 24. 10.
b 23. 10.
c *hyp. and sch* 22. 10
d 21. 10.
e 13. 10.
f *lem.* 60. 10.
g 2 *def.* 48. 10.

PROP. LXIII.

The square of a second bimedial line (AC + CB) *applied to a rational line* DE *makes the breadth* DG *a third binomial line*

As in the *prec* DL is ρ́ ⊓⊔ DE Furthermore because the rectangle ACB, and so LF (2 ACB) *a* is μν. *b* therefore shall LG be ρ́ '⊓⊔ DE. *c* Moreover DL '⊓⊔ LG. and also DL ⊓⊔ √ DLq − LGq. *d* therefore DG is a third binomial. *Which was to be dem.*

a *hyp and* 24 10.
b 23. 10.
c *lem.* 60. 10.
d 3 *def.* 48. 10.

PROP. LXIV.

The square of a Major line (AC + CB) *applied to a rational line* DE, *makes the breadth* DG *a fourth binomial line.*

a *hyp and* Again ACq + CBq. *i e.* DK *a* is $\dot{\rho}$*r. b* therefore DL is
*sch.*12.10. $\dot{\rho}$ ⊔ DE. also ACB, and so LF (2 ACB) *c* is *ur. d* there-
b 21. 10. fore LG is $\dot{\rho}$ ⊔ DE, *e* and consequently DL ⊔ LG.
c *hyp. and* Lastly because AC ⊔ BC. *f* shall DL be ⊔ DLq —
24. 10. LGq. *g* whence DG is a fourth binomial. *Which was*
d 23. 10. *to be dem.*
e 13. 10.
f *lem.* 60.
10.

PROP. LXV.

g 4. *def.*48. *The square of a line containing in power a rational and a*
10. *medial rectangle* (AC + CB) *applied to a rational line* DE *makes the latitude* DG *a fifth binomial.*

a 23. 10. Again, DK is μ. *a* therefore DL is $\dot{\rho}$ ⊔ DE. also
b 21. 10. LF is $\dot{\rho}$*r. b* therefore LG is $\dot{\rho}$ ⊔ DE. *c* therefore DL
c 13. 10. ⊔ LG. *d* likewise DL ⊔ √ DLq — LGq. *e* and so
d *lem.* 60. by consequence DG is a fifth binomial. *Which was to be*
10. *demonstrated.*
e 5. *def.*48.
10.

PROP. LXVI.

The square of a line containing in power two medial rectangles (AC + CB) *applied to a rational line* DE, *makes the latitude* DG *a sixth binomial line.*

As before, DL and LG are $\dot{\rho}$ ⊔ DE. But because
a *hyp.* ACq + CBq (DK) *a* ⊔ ACB, *b* and so DK ⊔ LF (2
b 14. 10. ACB) and also DK. LF *c* :: DL. LG. *d* therefore shall
c 1 6. DL be ⊔ LG. *e* Lastly DL ⊔ √ DLq — LGq. *f* by
d 10. 10. which it appears that DG is a sixth binomial.
e *lem.* 60.
10.

Lemma.

f 6. *def.*48.
10.

A——————C——————B   A————B————C
D——————————F——————E   D————————E————F

Let AB, DE *be* ⊔. *and make* AB. DE :: AC. DF.

I say 1. AC ⊔ DF. as appears by 10. 10. also CB ⊔
a 19 5. FE. *a* because AB. DE :: CB. FE.

2. AC CB :: DF. FE. For AC. DF :: AB. DE :: CB. FE. thereforefore by permutation AC. CB :: DF. FE

3. The

3. *The Rectangle* ACB ⊐⊏ DFE. For ACq. ACB *b* :: b 1. 6.
AC. CB *c* :: DF. EF :: DFq. DFE. wherefore by per- c *before*
mutation ACq. DFq :: ACB. DFE. therefore since ACq
⊐⊏ DFq. *d* shall ACB be ⊐⊏ DFE. d 10. 10.
4. ACq + CBq ⊐⊏ DFq + FEq. For because ACq.
CBq *e* :: DFq. FEq. therefore by compounding ACq. + e 22. 6.
CBq. CBq :: DFq + FEq. FEq. therefore since CBq ⊐⊏
FEq, *f* shall also ACq + CBq be ⊐⊏ DFq. + FEq. f 10. 10.
5 *Hence*, If AC ⊐⊐ or ⊐⊐ CB, *g* then likewise shall g 10. 10.
DE be ⊐⊐ or ⊐⊐ EF.

PROP. LXVII.

A line DE, *commensurable in length to a binomial line* (AC + CB) *is it self a binomial line, and of the same order.*

A———C———B
D————F——E

Make AB. DE :: AC. DF. *a* then are AC, DF ⊐⊏. *a* a *lem.* 66.
and CB, FE ⊐⊏. whence since AC and CB, *b* are ρ ⊐⊐, 10.
c thence DF, FE ρ ⊐⊐. therefore DF is a binomial. But b *hyp.*
because AC. CB *a* :: DF. FE. If AC ⊐⊏ or ⊐⊏ √ ACq c *lem.* 66.
— BCq, *d* then in like manner DF ⊐⊏ or ⊐⊏ √ DFq 10. *and*
— FEq also if AC ⊐⊏ or ⊐⊏ ρ propounded, *e* then *sch.* 12. 10.
shall DF be ⊐⊏ or ⊐⊏ ρ propounded. But if CB ⊐⊏ d 15. 10.
or ⊐⊏ ρ. likewise FE ⊐⊏ or ⊐⊐ ρ. If both AC, CB, e 12. 10.
⊐⊏ ρ. *f* then also both DF, FE, ⊐⊏ ρ. *g* That is, what- *and* 14 10.
soever binomial AB is, DE shall be of the same order. f *by def* 48.
Which was to be dem. 10.
g 14 10.

PROP. LXVIII.

A line DE *commensurabe in length to a bimedial line* (AC + CB) *is also a bimedial line, and of the same order*

Make AB. DE :: AC DF. *b* therefore AC ⊐⊏ DF. and a 12. 6.
CB ⊐⊏ FE. therefore seeing AC and CB *c* are μ, *d* also b *lem.* 66.
DF and FE shall be μ. and because AC *e* ⊐⊐ CB, *e* there- 10.
fore FD ⊐⊐ FE. *f* therefore DE is 2 μ. Wherefore if c *hyp.*
the rectangle ACB be ρ. because DFE *b* ⊐⊏ ACB, *g* d 24. 10.
likewise DFE is ρν and if that be μν, *h* this shall be μν e 10. 10.
too *k* That is, whether AB be 1 bimed. or 2 bimed. DF f 38. 10.
shall be of the same order. *Which was to be dem.* g *sch.* 12.
10
h 24. 10.
k 38. *or* 39.
10.

PROP. LXIX.

A———— C———B
D———— F————E

A line DE commensurable to a Major line (AC + CB) *is it self a Major line.*

a *hyp.* Make AB. DE :: AC. DF. Because AC *a* ⊔ CB, *b*
b *lem.* 66. thence DF ⊔ FE. also ACq + CBq *a* is ῥν. and so be-
10. cause DFq + FEq *b* ⊔ ACq + CBq, *c* also DFq +
c *sch.* 12 10 FEq is ῥν lastly, the rectangle ACB *a* is μν, *d* therefore
d 24. 10. the rectangle DFE is μν. (because DFE is *b* ⊔ ACB)
e 40. 10. *e* wherefore DE is *a Major* line. *Which was to be demonstrated.*

PROP. LXX.

A line DE commensurable to a line containing in power a rational and a medial rectangle (AC + CB) *is a line containing in power a rational and a medial rectangle.*

a *hyp.* Again make AB. DE :: AC. AC. DF. Because AC *a* ⊔
b *lem.* 66. CB, *b* also DF ⊔ FE. likewise because ACq + CBq *a* is
10. μν, *c* therefore DFq + FEq shalll be μν lastly because the
c 24. 10. rectangle ACB *a* is ῥν, *d* also DFE is ῥν. Therefore DE
d *sch.* 12. 10. contains in power ῥν and μν. *Which was to be dem.*
e 41. 10.

PROP. LXXI.

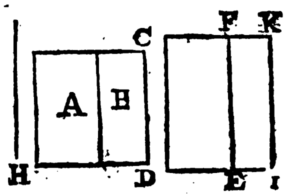
A———C———B
D———F———E

A line DE commensurable to a line containing two medial rectangles in power (AC + CB) *is also a line containing in power two medial rectangles.*

a *hyp.* Divide DE, as in the prec. Because ACq *a* ⊔ CBq,
b *lem.* 66. *b* thence shall DFq be ⊔ FFq, also because ACq +
10. CBq *a* is μν, *c* shall DFq + FEq be also μν. And in
c 24. 10. like manner because ACB *a* is μν, *d* also DFE is μν.
d 24. 10. Lastly, because ACq + CBq ⊔ ACB, *e* shall DFq +
e 14. 10. FEq be ⊔ DFE. *f* From whence it follows that DE
f 42 10. contains in power 2 μα. *Which was to be dem.*

PROP. LXXII.

If a rational rectangle A and a medial B, are compounded, four irrational lines will be made; either a binomial, or a first bimedial, or a Major, or a line containing in power a rational and a medial rectangle.

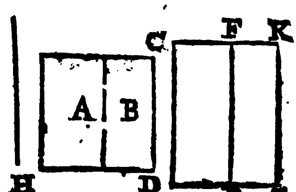


Namely, If Hq = A + B, then H shall be one of the four lines which the Theorem mentions. For upon CD the propounded ρ, *a* make the rectangle CE = A, a *cor.* 16. 6.
and FI = B *b* and so CI = Hq. Whereas then is A ρν, b 2. *ax.* 1.
likewise, CE is ρν. *c* therefore the latitude CF is ρ ⊏⊐ c 21. 10.
CD, and because B is μν, also FI shall be μν. *d* there- d 23. 10.
fore FK is ρ ⊏⊐ CD, *e* therefore CF, FK are ρ ⊏⊐. e 13. 10.
and so the whole CK *f* is binom. wherefore if A ⊏ B, f 37. 10.
i. e. CE ⊏ FI, *g* then CF ⊏ FK. therefore if CF ⊏⊐ g 1. 6.
√ CFq − FKq, *h* likewise CK shall be a 1 bin. and h 1. *def.* 48.
consequently H = √ CI *k* is a bin. If CF be supposed 10.
⊏⊐ √ CFq − FKq, *l* then shall CK be a 4 bin. where- k 55. 10.
fore H (√ CI) *m* is a major line. But if A ⊐ B, *g* then l 4. *def.* 48.
shall CF be ⊐ FK. consequently if FK ⊏⊐ √ FKq 10.
− CFq, *n* then shall CK be a 2. bin. *o* wherefore H is m 58. 10.
a first 2 μ. lastly if FK ⊏⊐ √ FKq − CFq, *p* then CK n 2. *def.* 48.
shall be a fifth binom. *q* whence H shall contain in pow- 10.
er ρν and μν. *Which was to be dem.* o 56. 10.

p 5. *def.* 48. 10.
q 59. 10.

PROP. LXXIII.

If two medial rectangles, A, B, incommensurable to one another are composed, two remaining irrational lines are made, either a second bimedial, or a line containing in power two medial rectangles.



As H containing in power A + B is one of the said irrational lines. For upon CD propounded ρ make the rectangle CE = A, and FI = B. whence Hq = CI.

Therefore because CE and FI *a* are $\mu\alpha$. *b* the latitudes
a *hyp.* CF, FK, shall be ρ ⌐⊔ CD. also because CE *a* ⌐⊔ FI,
b 23. 10. and CE. FI *c* :: CF. FK, *d* therefore CF ⌐⊔ FK. *e*
c 1 6. therefore CK is a 3 bin. namely, if CF ⊤⊔ $\surd$ CFq —
d 10. 10. FKq, whence H = $\surd$ CI *f* shall be 2 μ. But if CF
e 3. *def* 48. ⌐⊔ $\surd$ CFq — FKq, *g* then CK shall be a 6 binom. *h*
10. and consequently H contains in power 2 $\mu\alpha$. *Which was*
f 57. 10. *to be demonstrated.*
g 6 *def* 48.
10.
h 60. 10.

Here begins the Senaires of lines irrational by Subtraction.

PROP. LXXIV.

D———E———F *If from a rational line* DF *a rational line* DE, *commensurable in power only to the whole* DF, *be taken away, the residue* EF *is irrational, and is called an Apotome or residual line.*

a *lem.* 26. 10. For EFq *a* ⌐⊔ DEq; *b* but DEq is $\rho\nu$; *c* therefore EF is ρ. *Which was to be dem.*

b *hyp.* In numbers; let there be DF, 2. DE, $\surd$3. then EF
c 10 & 11. *def.* 10. shall be 2 — $\surd$ 3.

PROP. LXXV.

D———E———F *If from a medial line* DF *a medial line* DE *commensurable only in power to the whole* DF, *and comprehending with the whole* DF *a rational rectangle, be taken away, the remainder* EF *is irrational, and is called a first residual line of a medial*

a *sch.* 26. 10. For EFq *a* ⌐⊔ to the rectangle FDE. therefore seeing FDE *b* is $\rho\nu$. *c* EF shall be ρ. *Which was to be demonstrated.*

b *hyp.*
c 20. *and* 11. *def.* 10. In numbers, let DF be $v \surd$ 54, and DE $v \surd$ 24, therefore EF is $v \surd$ 54 — $v \surd$ 24.

PROP. LXXVI.

D———E———F *If from a medial line* DF, *a medial line* DE, *be taken away being commensurable only in power to the whole* DF, *and comprehending together with the whole line* DF *a medial rectangle, the remainder* EF *is irrational, and is called a second residual of a medial line.* Be-

Because DFq and DEq *a* are $\mu\alpha$ ⊓⊔, *b* therefore shall DFq + DEq be ⊓⊔ DEq. *c* wherefore DFq + DEq is $\mu\nu$. also the rectangle FDE, and so 2 FDE, *a* is $\mu\nu$. therefore EFq (*d* DFq + DEq — 2 FDE) *e* is $\rho\nu$. wherefore EF is ρ. *Which was to be dem.* — a *hyp.* b 16. 10. c 24. 10. d *cor.* 7. 2. e 27. 10.

In numbers, let DF be $v\sqrt{\ 18}$. and DE $v\sqrt{\ 8}$. then EF $v\sqrt{\ 18} - v\sqrt{\ 8}$.

PROP. LXXVII.

A ——— B —— C

If from a right line AC *be taken away a right line* AB *being incommensurable in power to the whole* BC, *and making with the whole* AC *that which is composed of their squares rational, and the rectangle contained under them medial, the remainder* BC *is irrational, and is called a Minor line.*

For ACq + ABq *a* is $\rho\nu$. but the rectangle ACB *a* is $\mu\nu$. *b* therefore 2 CAB ⊓⊔ ACq + ABq (*c* 2 CAB + BCq.) *d* therefore ACq + ABq ⊓⊔ BCq, *e* therefore BC is ρ. *Which was to be dem* — a *hyp* b *sch.* 12. 10. c 7. 2. d 17. 10. e 11. *def.* 10.

In numbers, let AC be $\sqrt{:\ 18 + \sqrt{\ 108}}$; AB $\sqrt{:\ 18 - \sqrt{\ 108}}$. then BC is $\sqrt{:\ 18 + \sqrt{:\ 108}} - \sqrt{:\ 18 - \sqrt{\ 108}}$.

PROP. LXXVIII.

D ——— E ——— F

If from a right line DF *be taken away a right line* DE, *being incommensurable in power to the whole line* DF, *and with the whole* DF *making that which is composed of their squares medial, and the rectangle contained under the same lines rational, the line remaining* EF *is irrational, and is called a line making a whole space medial with a rational space.*

For 2 FDE *a* is $\rho\nu$. *b* and DFq + DEq is $\mu\nu$. *c* therefore 2 FDE ⊓⊔ DFq + DEq *d* (2 FDE + EFq) *e* therefore EF is ρ. *Which was to be dem.* — a *hyp.* & *sch.* 12. 10. b *hyp.* c *sch.* 12. 10. d 7. 2. e *sch.* 12. 10. and 11 *def.* 10.

In numbers, let DF be $\sqrt{:\ \sqrt{\ 216} + \sqrt{\ 72}}$; DE $\sqrt{:\ \sqrt{\ 216} - \sqrt{\ 72}}$. therefore EF is $\sqrt{:\ \sqrt{\ 216} + \sqrt{\ 72}} - \sqrt{:\ \sqrt{\ 216} - \sqrt{\ 72}}$.

PROP. LXXIX.

D ——— E ——— F

If from a right line DF *be taken away a right line* DE, *incommensurable in power to the whole* DF, *and which toge-*

ther with the whole makes that which is composed of their squares medial, and the rectangle contained under them also medial and incommensurable to that which is composed of their squares, the remainder is irrational, and is called a line making a whole space medial with a medial space.

a hyp. & 24. 10. For 2 FDE, and FDq + DEq *a* are $\mu\alpha$; *b* therefore
b 27.10. EFq (*c* DFq + DEq − 2 FDE) is $\dot{\rho}\nu$. *d* and so consequently EF is $\dot{\rho}$. *Which was to be dem.*

c cor. 7 2. In numbers; let DF be $\sqrt{:\sqrt{180}+\sqrt{60}}$. DE $\sqrt{:}$
d 11 def 10. $\sqrt{180}-\sqrt{60}$. then EF shall be $\sqrt{:\sqrt{180}+\sqrt{60}}-\sqrt{:\sqrt{180}-\sqrt{60}}$.

Lemma.

B——M——G D——E——F
C—— H——

If there be the same excess between the first magnitude BG *and the second* C (MG) *as is between the third magnitude* DF *and the fourth* H (EF;) *then alternately, the same excess shall be between the first magnitude* BG *and the third* DF, *as is between the second* C *and the fourth* H.

a hyp. For because that *a* to the equals BM, DE, are added the equals MG, EF, that is, C, H; the excess of the
b 15. ax. 1. wholes BG, DF, *b* shall be equal to the excess of the parts added C, H. *Which was to be dem.*

Coroll.

Hence, Four magnitudes Arithmetically proportional, are alternately also Arithmetically proportional.

PROP. LXXX.

A——B——D——C

To an Apotome or residual line AB *only one rational right line* BC, *being commensurable in power only to the whole* AB, *is congruent, or can be joyned.*

If

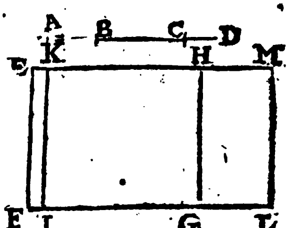
If it be possible, let some other line BD be added to
a 22. 10. it; *a* then the rectangles ACB, ADB, *b* and so conse-
b 22 10. quently the doubles of them are $\mu\alpha$. wherefore seeing
c *cor* 7. 2. ACq + BCq − 2 ACB *c* = ABq *c* = ADq + DBq − 2
d *lem* 79. ADB, therefore alternately ACq + BCq −: ADq +
10. BDq *d* = 2 ACB −: 2 ADB. But ACq + BCq −: ADq
e *hyp.* and +BDq *e* is $\rho\nu$ *f* therefore 2 ACB −: 2 ADB is $\rho\nu$. *Which*
27. 10. *is absurd.*
f *sch* 12. 10.
g 27. 10.

PROP. LXXXI.

A——B——D——C *To a first medial residual line* AB *only one medial right line* BC, *being commensurable only in power to the whole, and comprehending with the whole line a rational rectangle, can be joyned.*

Conceive BD to be such a line as may be joyned to
a *hyp.* it; then because ACq and BCq, as well as ADq and BDq
b 16. *and* *a* are $\mu\alpha$ $\neg\sqcup$. *b* also ACq + BCq, and ADq + BDq
24 10. shall be $\mu\alpha$. *c* but the rectangles ACB, ADB, *d* and so 2
c *hyp.* ACB and 2 ADB are $\rho\alpha$, *e* therefore 2 ACB −: 2 ADB,
d *sch* 12 10. *f* that is, ACq + BCq −: ADq + BDq is $\rho\nu$. *g Which*
e *sch.* 27. 10. *is absurd.*
f 7. 2. *and*
lem. 79. 10.
g 27. 10.

PROP. LXXXII.



Upon a second medial residual line AB *only one medial right line* BC, *commensurable only in power to the whole, and with it containing a medial rectangle, can be joyned.*

If it be possible, let some other line BD be ad-
a 4. 2 *and* ded to it; and upon EF ρ make the rectangle EG =
3 *ax.* 1. ACq + BCq; as also the rectangle EL = ADq + BDq.
b *hyp* likewise EI = ABq. Now 2 ACB + ABq = ACq +
c 24. 10. BCq = EG. therefore seeing EI = ABq, *a* also KG
d 23. 10. shall be = 2 ACB, moreover ACq and BCq *b* are $\mu\alpha$
e *hyp.* $\neg\sqcup$. *c* therefore EG (ACq + BCq) is $\mu\nu$. *d* therefore
f 24. 10. the breadth EH is ρ $\neg\sqcup$ EF. *e* Further, the rectangle
g *lem.* 26. ACB *f* and so 2 ACB (KG) is $\mu\nu$. *d* therefore KH is al-
10. so ρ $\neg\sqcup$ EF. Lastly, because ACq + BCq (EG) *g* $\neg\sqcup$
h 1. 6. 2 ACB (KG) and EG. KG *h* :: EH. KH. *k* therefore EH
k 10 10. $\neg\sqcup$ HK. *l* therefore EK is a residual line, whereto KH
l 74. 10. is

is congruent, by the same reason also shall KM be congruent to the said EK. *Which is repugnant to the* 80. prop. *of this Book.*

PROP. LXXXIII.

To a Minor line AB *only one right line* BC *can be joined being incommensurable in power to the whole, and making together with the whole line that which is composed of their squares rational, and the rectangle which is contained under them medial.*

A — B —— D — C

a 1. *hyp.*
b *lem* 97. 10.
c *sch* 27. 10.
d 27. 10.

Conceive any other BD to be congruent to it; Therefore whereas ACq + BCq, and ADq + BDq *a* are ῥα. their excess (2 *b* ACB —: 2 ADB) *c* is ῥν. *Which is absurd*; because ACB and ADB are μα by the Hyp.

PROP. LXXXIV.

Unto a line (AB) *making with a rational space a whole space medial only one right line* BC *can be joined, being incommensurable in power to the whole, and making together with the whole that which is composed of their squares medial, and the rectangle which is contained under them rational.*

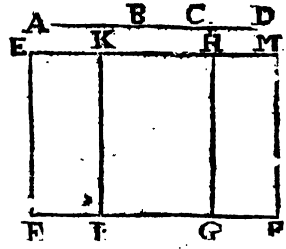
A — B —— D — C

a *hyp.*
b *sch* 12 10.
c *lem.* 79. 10.
d *sch.* 27. 10.

Suppose some other BD to be congruent also to it; *a* then the rectangles ACB, ADB, *b* and so 2 ACB and 2 ADB are ῥα. therefore 2 ACB —: 2 ADB, *c* that is, ACq + BCq —: ADq + BDq *d* is ῥν. *Which is absurd*; since ACq + BCq, and ADq + BDq are μα by the Hyp.

PROP.

PROP. LXXXV.



To a line AB, which with a medial space makes a whole space medial, can be joined only one right line BC, incommensurable in power to the whole, and making with the whole both that which is composed of their squares medial, and the rectangle which is contained under them medial and incommensurable to that which is composed of their squares.

Those things being supposed which are done and shewn in the 82 *prop.* of this Book; it is clear that EH and KH are ρ́ ⊐ EF. Besides, since ACq + CBq, that is, the rectangle EG. *a* is ⊐ ACB, *b* and so EG ⊐ 2 ACB (KG;) and EG. KG *c* :: EH. KH; shall EH be ⊐ KH. therefore EK is a residual line, and the line congruent to it is KH. In like manner may KM be shewn to be congruent to the residual EK, against the 80. *prop.* of this Book.

a *hyp.*
b 14. 10.
c 1. 6.

Third Definitions.

A Rational line and a residual being propounded, if the whole be more in power than the line joined to the residual, by the square of a right line commensurable unto it in length; then

I. If the whole be commensurable in length to the rational line propounded, it is called a first residual line.

II. But if the line adjoined be commensurable in length to the rational line propounded, it is called a second residual line.

III. If neither the whole nor the line adjoined be commensurable in length to the rational line propounded, it is called a third residual line.

Moreover, if the whole be more in power than the line adjoined by the square of a right line incommensurable to it in length, then

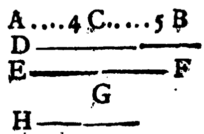
IV. If the whole be commensurable in length to the rational line propounded, it is called a fourth residual line.

V. But

V. But if the line adjoined be commensurable in length to the rational line propounded, it is a fifth residual.

VI. If neither the whole nor the line adjoined be commensurable in length to the rational line propounded, it is termed a sixth residual line.

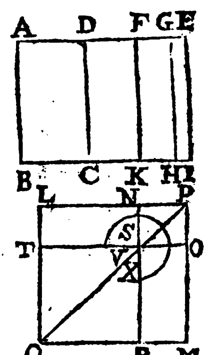
PROP. LXXXVI, 87, 88, 89, 90, 91.



To find out a first, second, third, fourth, fifth, and sixth residual line.

Residual lines are found out by subducting the less names or parts of binomials from the greater. Ex. gr. Let $6 + \sqrt{20}$ be a first binomial, then shall $6 - \sqrt{20}$ be a first residual. So that it is not necessary to repeat more concerning the finding of them out.

Lemma.



Let AC *be a rectangle contained under the right lines* AB, AD. *Let* AD *be drawn forth to* E, *and* DE *equally divided in* F. *and let the rectangle* AGE *be* = FEq *and the rectangles* AI, DK, FH, *finished. Then let the square* LM = AH *be made, and square* NO = GI; *and the lines* NSR, OST, *produced.*

I say, 1. The rectangle AI = LM + NO = TOq + SOq, which appears by the constr.

2. *The rectangle* DK = LO. For because the rectangle AGE *a* = FEq. *b* thence are AG, FE, GE ∺ *c* and so AH, FI, GI ∺, *a* that is, LM, FI, NO, ∺; but LM, LO, NO *d* are ∺; therefore FI = *e* LO *f* = DK = *g* NM.

3. *Hence,* AC = AI − DK − FI = LM + NO − LO − NM = TR.

4. *It is manifest that* DF, FE, DE, *are* ⊤⊥.

5. *If* AE ⊤⊥ DE, *and* AE ⊤⊥ $\sqrt{AEq - DEq}$, k *then shall* AG, GE, AE *be* ⊤⊥.

6. *Also, because* AE ⌊ ⊤⊥ DE, m *thence shall* AE, FE, *be* ⊤⊥. n *and so* AI, FI, *that is,* LM + NO *and* LO *are* ⊤⊥.

7. *Because*

a *constr.*
b 17. 6.
c 1. 6.
d *schol* 22. 6.
e 9. 5.
f 36. 1.
g 43. 1.
h 16. 10.
k 18. *and* 10. 10.
l *hyp.*
m 13. 10.
n 1. 6 & 10. 10.

* before. 7. *Because* AG * ⊓ GE, n *shall* AH, GI, *that is*, LM NO *be* ⊓.

o 14. 10. 8 *But because* AE l ⊓ DE, o *therefore shall* FE, GE, *be* ⊓, n *and so the rectangle* FI ⊓ GI, *that is*, LO ⊓

p 2. 6. NO. *wherefore seeing* LO. NO p:: TS. SO. q *therefore*

q 10. 10. *shall* TS, SO *be* ⊓.

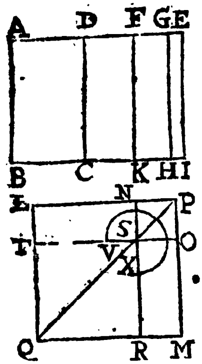
r 19. 10. & 9. *If* AE *be put* ⊓ √ AEq – DEq, *then shall* AG,

17. 10. GE, AE *be* ⊓.

s 1. 6. *and* 10 s *Wherefore the rectangles* AH, GI, *that is*, TOq,

10. 10. SOq *shall be* ⊓.

PROP. XCII.



If a space AC *be contained under a rational line* AB, *and a first residual line* AD (AE – DE) *the right line* TS, *which containeth the space* AC *in power, is a residual line.*

Use the foregoing *Lemma* for a preparation to the demonstration of this prop. Therefore TS = √ AC. Also AG, GE, AE, are

a hyp. ⊓; therefore since AE ⊓ a AB

b 12. 10. ῥ, b also AG and GE shall be ⊓

c 20. 10. AB. c therefore the rectangles AH

d lem. 91. and GI, that is, TOq and SOq are

10. ῥα. d Likewise TO, SO, are ῥ ⊓

e 74. 10. e and consequently TS is a residual line. *Which was to be demonstrated.*

PROP. XCIII.

See the prec. Scheme.

If a space AC *be contained under a rational line* AB *and a second residual* AD (AE – DE) *the right line* TS, *containing the space* AC *in power, is a first medial residual line.*

a hyp. Again, by the foregoing *Lemma*, AG, GE, AE are

b 1 . 10. ⊓. therefore a since AE is ῥ ⊓ AB, b also AG, GE,

c 22. 10. shall be ῥ ⊓ AB, c therefore the rectangles AH, GI,

d lem. 74. that is, TOq, SOq are μα. d likewise TO ⊓ SO. Lastly,

10. because DE e ⊓ AB ῥ, f the right angle DI, and the

e hyp. half thereof DK or LO, that is, TOS shall be ῥα g

f 20 10. from whence it follows that TS (√ AC) is a first medial

g 75. 10. residual. *Which was to be dem.* PROP.

PROP. XCIV.

See Scheme 92.

If a space AC *be contained under a rational line* AB *and a third residual* AD (AE − DE) *the right line* TS *containing in power the space* AC *is a second medial residual line.*

As in the former, TO and SO are μ. Therefore be-
a hyp. cause DE *a* is $\dot{\rho}$ ⊓ AB, *b* the rectangle DI, *c* and so DK,
b 22. 10. or TOS, shall be $\mu\nu$. therefore TS = √ AC is a second
c 24. 10. medial residual. *Which was to be dem.*
d 76. 10.

PROP. XCV.

See Scheme 92.

If a space AC *be contained under a rational line* AB *and a fourth residual* AD (AE — DE) *the right line* TS *containing the space* AC *in power, is a Minor line.*

a lem 91. 10 As before, TO *a* ⊓ SO. Therefore because AE *b* is
b hyp. $\dot{\rho}$ ⊓ AB, *c* shall AI (TOq + SOq) be $\dot{\rho}\nu$ but, as be-
c 20. 10. fore, the rectangle TOS is $\mu\nu$. *d* therefore TS = √ AC
d 77. 10. is a *Minor* line. *Which was to be dem.*

PROP. XCVI.

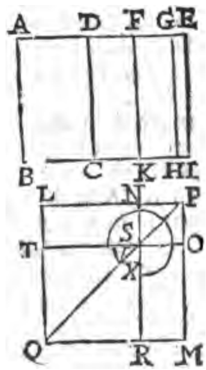
See Scheme 92.

If a space AC *be contained under a rational line* AB *and a fifth residual* AD (AE — DE) *the right line* TS *containing in power the space* AC; *is a line which maketh with a rational space the whole space medial.*

a hyp. For again TO ⊓ SO. therefore since AE *a* is $\dot{\rho}$ ⊓
b 22. 10. AB *b* also AI, that is, TOq + SOq shall be $\mu\nu$. But, as
c 78. 10. in the 93 the rectangle TOS is $\dot{\rho}\nu$. *c* whence TS = √
AC is a line which with $\dot{\rho}\nu$ makes a whole $\mu\nu$. *Which was to be dem.*

PROP.

PROP. XCVII.

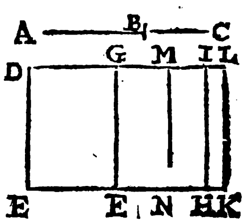


If a space AC *be contained under a rational line* AB, *and a sixth residual* AD (AE — DE) *the right line* TS *containing in power the space* AC *is a line making with a medial rectangle, a whole space medial.*

As often above, TO ⌐⊓ SO. also, as in 96. TOq + SOq is $\mu\nu$. but the rectangle TOS is $\dot{\rho}\nu$. as in 94. *a* Lastly, TOq + SOq ⊤⊓ TOS. *b* therefore TS = √ AC is a line which with $\mu\nu$ makes a whole $\mu\nu$. *Which was to be demonstrated.*

a *lem.* 91. 10.
b 79. 10.

Lemma.



Upon a right line DE * *apply the rectangles* DF = ABq, *and* DH = ACq, *and* IK = BCq. *and let* GL *be bisected in* M, *and the line* MN *drawn parallel to* GF. * *cor.* 16. 6.

Then 1. *The rectangle* DK *is* = ACq + BCq. *as the construction manifests*

2. *The rectangle* ACB = GN or MK. For DK *a* = ACq + BCq *b* = 2 ACB + ABq. but ABq *a* = DF. therefore GK *c* = 2 ACB. and consequently GN or MK = ACB.

3. *The rectangle* DIL = MLq. For because ACq. ACB *e* :: ACB, BCq, that is, DH MK :: MK. IK. *e* thence is DI ML :: ML. IL, *f* therefore DIL = MLq.

4. *If* AC *be taken* ⌐⊓ BC, *then* DK *shall be* ⊤⊓ ACq. For ACq + BCq (DK) *g* ⊤⊓ ACq.

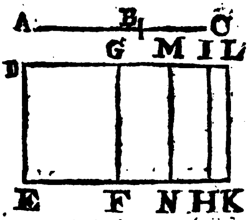
5. *Likewise* DL ⊤⊓ √ DLq — GLq. For because DH (ACq) ⊤⊓ IK (BCq) *h* thence shall DI be ⊤⊓ IL. *k* therefore √ DLq — GLq ⊤⊓ DL.

6 *Also* DL ⊤⊓ GL. For ACq + BCq ⊤⊓ *l* 2 ACB. that is, DK ⊤⊓ GK *m* therefore DL ⊤⊓ GL.

7. *But if* AC *be taken* ⌐⊓ BC, *then* DL shall be ⊤⊓ √ DLq — GLq.

a *constr.*
b 7. 2.
c 3. *ax.* 1.
d 7. *ax.* 1.
e 1. 6.
f 17. 6.
g 16. 10.
h 10. 10.
k 18. 10.
l *lem* 26 10.
m 10. 10.
n 19. 10.

PROP. XCVIII.



The square of a residual line AB (AC − BC) *applyed to a rational line* DE, *makes the breadth* DG *a first residual line*.

Do as is enjoined in the *Lemma* next preceding. Then because AC, BC, *a* are ρ̂ ⊤⊏ *b* also DK (ACq + BCq) shall be ⊐⊏ ACq *c* therefore DK is ρν. *d* wherefore DL is ρ̂ ⊐⊏ DE. *e* Likewise the rectangle GK (2 ACB) is ·ν. *f* therefore GL is ρ̂ ⊤⊏ DE, *g* and consequently DL ⊤⊏ GL. *h* But DLq ⊐⊏ GLq, *k* therefore DG is a residual, *l* and that of the first order (because *m* AC ⊤⊏ BC, and therefore DL ⊐⊏ √ DLq − GLq.) *Which was to be dem.*

a *hyp.* b *lem.* 97. 10. c *sch.* 12. 10. d 21. 10. e 22. *and* 24. 10. f 23. 10. g 13. 10. h *sch.* 12. 10. k 74. 10. l 1. *def.* 85. 10. m *lem.* 97. 10.

PROP. XCIX.

See the following Scheme.

The square of a first medial residual line AB (AC − BC) *applied to a rational line* DE, *makes the breadth* DG *a second residual line.*

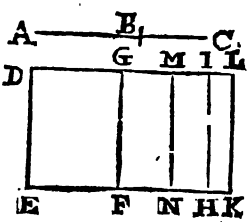
Supposing the foregoing *Lemma*; because AC and BC *a* are μ ⊤⊏, *b* thence shall DK (ACq + BCq) be ⊐⊏ ACq. *c* wherefore DK is μν. *d* therefore DL is ρ̂ ⊐⊏ DE. *e* also GK (2 ACB) is ρ̂ν. *f* therefore GL is ρ̂ ⊤⊏ DE; *g* wherefore DL ⊤⊏ GL. *h* But DLq ⊐⊏ GLq. *k* therefore DG is a residual line: And because DL is ⊐⊏ √ DLq − GLq, *m* therefore shall DG be a second residual. *Which was to be dem.*

a *hyp.* b *lem.* 97. 10. c 24. 10. d 23. 10. e *hyp. and sch.* 12. 10. f 21. 10. g 13. 10. h *sch.* 12. 10. k 74. 10. l *lem.* 97. 10. m 2. *def.* 85. 10.

PROP.

PROP. C.

The square of a second medial residual line AB (AC — BC) *applied to a rational line* DE, *makes the breadth* DG *a third residual line.*



Again DK is μν. *a* wherefore DL is ρ́ ˈ⊔⊔ DE. alſo GK is μν, *a* whence GL is ρ́ ˈ⊔⊔ DE. *b* likewiſe DK ˈ⊔⊔ GK. *c* wherefore DL ˈ⊔⊔ GL. *d* but DLq ⊔⊔ GLq. *e* therefore DG is a reſidual line, and that of *f* the third order, *g* becauſe DL ⊔⊔ √ DLq — GLq. *Which was to be dem.*

a 23. 10. b *lem.* 26. 10. c 1. 6. & 10. 10. d *ſch.* 12. 10 e 74. 10. f 3. *def.* 85. 10. g *lem.* 97. 10.

PROP. CI.

See the foregoing Scheme

The ſquare of a Minor line AB (AC — BC) *applied to a rational line* DE, *makes the breadth* DG *a fourth reſidual.*

As before, ACq + BCq, that is DK, is ρ́ν. *a* therefore DL is ρ́ ⊔⊔ DE. but the rectangle ACB, and ſo GK (2 ACB) * is μν. *b* wherefore GL is ρ́ ⊔⊔ DE. *c* therefore DL ˈ⊔⊔ GL. *d* but DLq ⊔⊔ GLq. and becauſe * ACq ˈ⊔⊔ BCq, *e* thence ſhall DL be ˈ⊔⊔ √ DLq — GLq. *f* therefore DG has the conditions required to a fourth reſidual. *Which was to be demonſtrated.*

a 21. 10. * *hyp.* b 23. 10. c 13. 10. d *ſch.* 12. 10. e *lem.* 97. 10. f 4. *def.* 85. 10.

PROP. CII.

See the foregoing Scheme.

The ſquare of a line AB (AC — BC) *which makes with a rational ſpace the whole ſpace medial, applied to a rational line* DE, *makes the breadth* DG *a fifth reſidual line.*

For, as above, DK is μν. *a* wherefore DL is ρ́ ˈ⊔⊔ DE alſo GK is ρ́. *b* whence GL is ρ́ ⊔⊔ DE. *c* therefore DL ˈ⊔⊔ GL. *d* but DLq ⊔⊔ GLq. Moreover DL *e* ˈ⊔⊔ √ DLq — GLq. wherefore DG *f* is a fifth reſidual. *Which was to be dem.*

a 23. 10. b 21. 10. c 13. 10. d *ſch.* 12. 10. e *lem.* 97. 10. f 5. *def.* 85. 10.

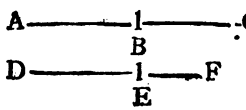
PROP. CIII.

See the same Scheme as before.

The square of a line AB (AC — BC) *making with a medial space the whole space medial, applied to a rational line* DE, *makes the breadth* DG *a sixth residual line.*

a 25. 10. As above, DK and GK are $\mu\alpha$; *a* wherefore DL
b *hyp. and* and GL are $\stackrel{\circ}{\rho}$ ⊏⊐ DE. also DK *b* ⊏⊐ GK. *c* whence
lem. 97. 10. DL ⊏⊐ GL. *d* therefore DG is a residual. *b* And where-
c 10. 10. as ACq ⊏⊐ BCq. and so DL ⊏⊐ $\surd$ DLq — GLq, *e*
d 74. 10. therefore DG shall be a sixth residual. *Which was to*
e 6. *def.* 85. *be demonstrated.*
10.

PROP. CIV.



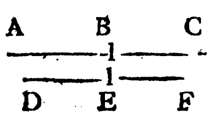
A right line DE *commensurable in length to a residual* AB (AC — BC) *is it self also a residual, and of the same order.*

Lemma.

Let AB. DE :: AC. DF, *and* AB ⊏⊐ DE.

I say AC + BC ⊏⊐ DF + EF. For AC. BC *a* :: DF. EF. therefore by compounding AC + BC. BC :: DF
+ EF. EF therefore by permutation AC + BC. DF
a *lem.* 66. + EF :: BC. EF. *a* but BC ⊏⊐ EF. *b* therefore AC +
10. BC ⊏⊐ DF + EF. *Which was to be dem.*
b 10. 10. *a* Make AB. DE :: AC. DF, *b* therefore AC + BC
a 12. 6. ⊏⊐ DF + EF. therefore seeing AC + BC *c* is a bi-
b *lem.* 103. nomial, *d* DF + EF shall be a binomial too, and of
10. the same order. *e* wherefore DF — EF is a residual of
c *hyp.* the same order with AC — BC. *Which was to be de-*
d 67. 10. *monstrated.*
e *by def* 85.
10.

PROP. CV.



A right line DE *commensurable to a medial residual line* AB (AC — BC) *is it self a medial residual, and of the same order.*

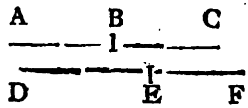
Again,

Again *a* make AB. DE :: AC. DF. *b* whence AC + BC $\sqcap\!\!\sqcup$ DF + EF. *c* therefore DF + EF is a bimedial of the same order with AC + BC, *d* and consequently DF — EF shall be a medial residual of the same order with AC — BC. *Which was to be dem.*

a 12. 6. b *lem.* 103. 10. c 68. 10. d 75. *and* 76. 10.

PROP. CVI.

A right line DE *commensurable to a Minor line* AB (AC — BC) *is it self also a Minor line.*

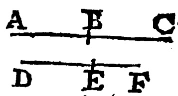


Make AB. DE :: AC. DF. *a* then is AC + BC $\sqcap\!\!\sqcup$ DF + EF. but AC + BC *b* is a *Major* line; *c* therefore DF + EF is also a *Major* line; *d* and consequently DF — EF is a *Minor* line. *Which was to be dem.*

a *lem.* 103. 10. b *hyp.* c 69. 10. d 77. 10.

PROP. CVII.

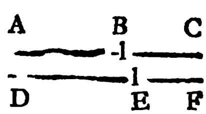
A right line DE *commensurable to a line* AB (AC — BC) *which makes with a rational space the whole space medial, is it self also a line making with a rational space the whole space medial.*



For, accordingly as in the former, we may shew DF + EF to contain in power $\rho\nu$ and $\mu\nu$. *a* whence DF — EF is a line making, &c. a 78. 10.

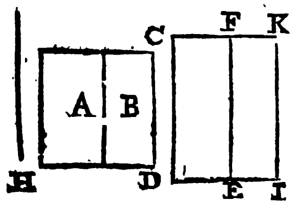
PROP. CVIII.

A right line DE *commensurable to a line* AB (AC — BC) *which with a medial space makes the whole space medial, is it self a line making with a medial space the whole space medial.*



For according to the preceding DF + EF shall contain in power 2 $\mu\alpha$. *a* therefore DF — EF shall be, as in the Prop. a 79. 10.

PROP. CIX.



A medial rectangle B *being taken from a rational rectangle* A + B, *the right line* H *which containeth in power the space remaining* A, *is one of these two irrational lines,* viz. *either a residual line, or a Minor line.*

Upon CD ῥ make the rectangles CI = A + B, and FI = B. whence CE *a* = A = Hq. wherefore because CI *b* is ῥν. *c* therefore CK is ῥ ⊐⊏ CD. but because FI *b* is μν, *d* shall FK be ῥ ⊐⊏ CD. *e* whence CK ⊐⊏ FK. *f* therefore CF is a residual line. Wherefore if CK be ⊐⊏ √ CKq − FKq, *g* then CF shall be a first residual. *h* therefore √ CE (H) is a residual line. But if CK ⊐⊏ √ CKq − FKq *k* then CF shall be a fifth residual; and consequently H (√ CE) *l* shall be a *Minor* line. *Which was to be dem.*

a 3. *ax.* 1.
b *hyp. and constr.*
c 21. 10.
d 23. 10.
e 13. 10.
f 74. 10.
g 1. *def.* 85. 10.
92. 10.
k 4. *def.* 85. 10.
l 95. 10.

PROP. CX.

See the preceding Scheme.

A rational rectangle B *being taken away from a medial rectangle* A+B, *other two irrational lines are made, namely; either a first medial residual line, or a line making with a rational space the whole space medial.*

Upon CD the propounded ῥ make the rectangles CI = A + B, and FI = B *a* whence CE = A = Hq. Therefore because CI *b* is μν; *c* shall CK be ῥ ⊐⊏ CD. but because FI *b* is ῥν. *d* thence FK ῥ ⊐⊏ CD. *e* whence CK ⊐⊏ FK. *f* therefore CF is a residual, *g* and that a second. If CK ⊐⊏ √ CKq − FKq, *h* then H (√ CE) is a first medial residual. But if CK ⊐⊏ √ CKq − FKq, *k* then shall CF be a fifth residual; and *l* consequently H (√ CE) shall be a line making μν with ῥν. *Which was to be dem.*

a 3. *ax.* 1.
b *hyp. and constr.*
c 23. 10.
d 21. 10.
e 13. 10.
f 74. 10.
g 2. *def.* 85. 10.
h 93. 10.
k 5 *def.* 85. 10.
l 96. 10.

PROP.

PROP. CXI.

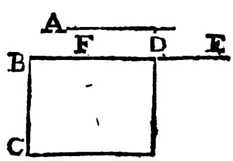
See the same Scheme.

A medial space B *being taken away from a medial space* A + B, *which is incommensurable to the whole* A + B, *the other two irrational lines are made*, viz. *either a second medial residual line, or a line making with a medial space the whole space medial.*

Upon CD ρ́ make the rectangles CI = A + B, and FI = B. *a* wherefore CE = A = Hq. Because therefore CI is μν, *b* thence CK is ρ́ ⊥ CD. and in like manner FK ρ́ ⊥ CD. Likewise because CI *c* ⊥ FI, *d* therefore CK ⊥ FK. *e* wherefore CF is a residual, *f* namely a third, If CK ⊥ √ CKq — FKq, *g* whence H (√ CE) shall be a second medial residual, but if CK ⊥ √ CKq − FKq. *h* then shall CF be a sixth residual. *k* wherefore A shall be a line making μν with μ. *Which was to be dem.*

a 3. *ax.* 1.
b 23. 10.
c *hyp.*
d 10. 10.
e 74. 10.
f 3 *def.* 85. 10.
g 94. 10.
h 6. *def.* 85. 10.
k 97. 10.

PROP. CXII.



A residual line A *is not the same with a binomial line*

Upon BC propounded ρ́ make the rectangle CD = Aq. Therefore seeing A is a residual, *a* BD shall be a first residual, to which let DE be the line congruent or that may be adjoined. *b* wherefore BE, DE, are ρ́ ⊥. *c* and BE ⊥ BC. If you conceive A to be a binomial, then BD is a first binomial, whose names let be BF, FD; and let BF be ⊏ FD. *d* therefore BF, FD are ρ́ ⊥; and BF *e* ⊥ BC. therefore since BC ⊥ BE, *f* shall BE be ⊥ BF. *g* and thence BE ⊥ FE. *h* therefore FE is ρ́. Likewise because BE ⊥ DE, *k* shall FE be ⊥ DE. *l* wherefore FD is a residual, and so FD is ρ́. but it was shewn ρ́. which are repugnant. therefore A is falsely conceived to be a binomial. *Which was to be dem.*

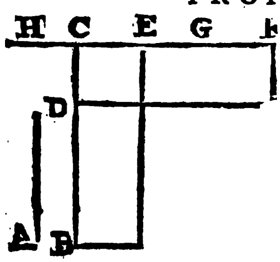
a 98. 10.
b 74. 10.
c 1. *def.* 85. 10.
d 37. 10.
e 1. *def.* 48. 10.
f 12. 10.
g *cor.* 16. 10.
h *sch.* 12. 10.
k 14. 10.
l 74. 10.

The names of the 13 irrational lines differing one from another.

1. A Medial line.
2. A binomial line; of which there are six species.
3. A first bimedial line.
4. A second bimedial.
5. A Major line.
6. A line containing in power a rational superficies, and a medial superficies.
7. A line containing in power two medial superficies.
8. A residual line; of which there are also six kinds.
9. A first medial residual line.
10. A second medial residual line.
11. A Minor line.
12. A line making with a rational superficies the whole superficies medial.
13. A line making with a medial superficies the whole superficies medial.

Since the differences of breadths do argue differences of right lines, whose squares are applied to some rational line, and it is demonstrated in the preced. Propositions that the breadths which arise from applying of the squares of these 13 lines do differ one from another, it evidently follows that these 13 lines do also differ one from another.

PROP. CXIII.



The square of a rational line A *applied to a binomial* BC (BD + DC) *makes the breadth* EC *a residual line, whose names* EH, CH, *are commensurable to the names* BD, DC, *of the binomial line; and in the same proportion* (EH. BD :: CH. DC;) *and moreover, the residual line* EC *which is made, is of the same order with* BC *the binomial.*

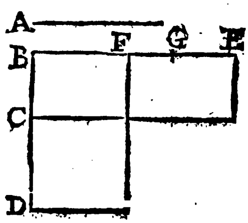
a *cor.* 16. 6. Upon DC the less name *a* make the rectangle DF = Aq = BE, whence BC. CD *b* :: FC. CE. there-
b 14. 6. fore by division, BD. DC :: FE. EC. And whereas
c *hyp.* BD *c* ⊏ DC, *d* thence FE shall be ⊏ EC, Take EG
d 14. 5. = EC, and make FG. GE :: EC. CH. Then EH, and CH shall be the names of the residual EC, where-

unto

unto all is agreeable that is propounded in the theorem.
For by compounding, FE. GE (EC) :: EH. CH.
therefore FH. EH *e* :: EH. CH *f* :: FE. EC *f* :: BD. e 12. 5.
DC. wherefore since BD *g* ⊐⊏ DC *h* thence shall EH f *before.*
be ⊐⊏ CH, *h* and FHq ⊐⊏ EHq Therefore because g *hyp.*
FHq. EHq *k* :: FH. CH. *h* shall FH be ⊐⊏ CH. *l* and h 10. 10.
so FC ⊐⊏ CH. Moreover CD *g* is ρ, and DF (Aq) k *cor.* 20. 6.
g is ρν. *m* therefore FC is ρ, ⊐⊏ CD. whence also CH l 16. 10.
is ρ ⊐⊏ CD. *n* therefore EH, CH are ρ and ⊐⊏, as be- m 21. 10.
fore. *o* therefore EC is residual line, to which CH may n *sch.* 12.
be joined. Furthermore EH. CH *f* :: BD. DC, and so by 10.
permutation EH. BD :: CH. DC. whence because CH *f* o 74. 10.
⊐⊏ DC, *p* shall EH be ⊐⊏ BD. But suppose BD ⊐⊏ p 10. 10.
√ BDq — DCq, *q* then shall EH be ⊐⊏ √ EHq. — q 15. 10.
CHq. Also if BD ⊐⊏ ρ propounded, then shall EH r 12. 10.
be ⊐⊏ to the same ρ. *s* that is, if BC be a first bino- s 1. *def* 48.
mial, *t* EC shall be a first residual. In like manner, if 10.
DC be to the ⊐⊏ propounded ρ, *r* then is CH ⊐⊏ to t 1. *def.* 85.
the same ρ. *u* that is, if BC be a second binomial, *x* 10.
EC shall be a second residual: And if this be a third u 2. *def.* 48.
binomial, then that shall be a third residual, &c. But 10.
if BD be ⊐⊏ √ BDq — DCq, *y* then shall EH be ⊐⊏ x 2. *def.* 85.
√ EHq — CHq. therefore if BC be a 4, 5, or 6 bi- 10.
nomial, EG shall be likewise a 4, 5, or 6 residual. y 15. 10.
Which was to be dem.

PROP. CXIV.

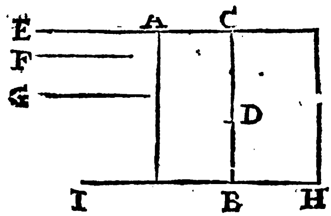
The square of a rational line A *applied to a residual line* BC (BD—CD) *makes the breadth* BE *a binomial; whose names* BE, GE *are commensurable to the names* BD, BC *of the residual line* BC, *and in the same proportion, and moreover, the binomial line which is made* (BE) *is of the same order with the residual line* (BC.)



a Make the rectangle DF = Aq. and BF. FE *b* a *cor.* 16. 6.
:: EG. GF. whence for that DF = Aq = CE, *c* there- b 12. 6.
fore BD. BC :: BE. BF. therefore by conversion of pro- c 14. 6.
portion BD. CD :: BE. FE :: EG. GF :: *d* BG. EG. d 19. 5.
but BD *e* ⊐⊏ CD. *f* therefore BG ⊐⊏ GE. therefore e *hyp.*
be- f 10. 10.

g *cor.*20. 6. becauſe BGq. GEq *g* :: BG. GF. *h* ſhall BG be ⊐⊏ GF.
h 10. 10. *k* and ſo BG ⊐⊏ BF. moreover BD *e* is $\acute{\rho}$, and the rec-
k *cor.*16.10. tangle DF (Aq) *e* is $\acute{\rho}v$. *l* therefore BF is $\acute{\rho}$ ⊐⊏ BD. *m*
l 21. 10. therefore alſo BG is $\acute{\rho}$ ⊐⊏ BD. *n* therefore BG, GE
m 12. 10. are $\acute{\rho}$ ⊐⊏. *o* wherefore BE is a binomial. Laſtly, be-
n *ſch.*12 10. cauſe BD. CD :: BG. GE. and by permutation BD. BG ::
o 37. 10. CD. GE, and BD ⊐⊏ BG. *p* thence ſhall CD be ⊐⊏ GE.
p 10. 10. therefore if CB be a firſt reſidual, BE ſhall be a firſt binomial, &c. as in the *prec.* therefore, &c.

PROP. CXV.



If a ſpace AB *be contained under a reſidual line* AC (CE — AE) *and a binomial* CB, *whoſe names* CD, DB *are commenſurable to the names* CE, AE, *of the reſidual line, and in the ſame proportion* (CE. AE :: CD. DB) *then the right line* F *which containeth in power that ſpace* AB, *is rational.*

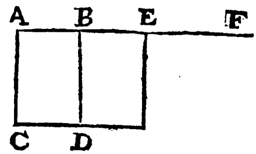
Let G be $\acute{\rho}$. and make the rectangle CH = Gq; *a*
a 113. 10. then ſhall BH (HI — IB) be a reſidual line, and HI *a*
b *hyp.* ⊐⊏ CD *b* ⊐⊏ CE. *a* and BI ⊐⊏ DB. *a* and HI. BI :: CD.
c 19 5. DB *b* :: CE. EA. therefore by permutation HI. CE ::
d 12. 10. BI. EA. *c* therefore BH. AC :: HI. CE :: BI. EA:
e 10. 10. wherefore ſince *d* HI ⊐⊏ CE, *e* thence BH ⊐⊏ AC.
f 1. 6. *and* 10. 10. *f* therefore the rectangle HC ⊐⊏ BA. But HC (Gq) *b* is $\acute{\rho}v$. *g* therefore BA (Fq) is $\acute{\rho}v$. and conſequently F
g *ſch.* 12. 10. is $\acute{\rho}$. *Which was to be dem.*

Coroll.

Hereby it appears that a rational ſuperficies may be contained under two irrational right lines.

PROP.

PROP. CXVI.

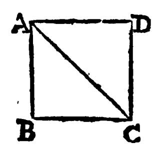


Of a medial line AB *are produced infinite irrational lines* BE, EF, &c. *whereof none is of the same kind with any of the precedent.*

Let AC be propounded $\acute{\rho}$. and AD a rectangle contained under AC, AB. *a* therefore AD is $\acute{\rho}\nu$. Take BE $= \sqrt{}$ AD. *b* then BE is $\acute{\rho}$, and the same with none of the former. For no square of any of the former being applied to $\acute{\rho}$, makes the breadth medial. Let the rectangle DE be finished, *a* then DE shall be $\acute{\rho}\nu$. and *b* consequently EF ($\sqrt{}$ DE) shall be $\acute{\rho}$, and not the same with any of the former, for no square of the former being applied to $\acute{\rho}$, makes the latitude BE. therefore, &c.

a *lem.* 38. 10.
b 11. 10.

PROP. CXVII.



Let it be required to shew that in square figures BD, *the diameter* AC *is incommensurable in length to the side* AB.

For ACq. ABq *a* :: 2. 1 *b* :: not Q. Q. *c* therefore AC ⊥⊏ AB. *Which was to be dem.* This Theorem was of great note with the ancient Philosophers; so that he that understood it not was esteemed by *Plato* undeserving the name of a man, but rather to be reckoned among brutes.

a 47. 1.
b *cor.* 24. 8.
c 9. 10.

The End of the Tenth Book.

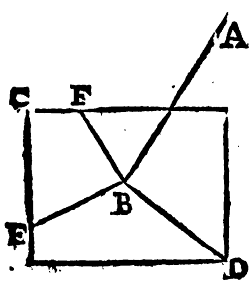
THE

The ELEVENTH BOOK OF EUCLIDE's ELEMENTS.

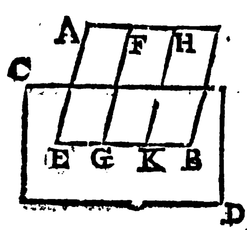
Definitions.

I. A Solid is that which hath length, breadth and thickness.

II. The term, or extreme of a solid is a Superficies.



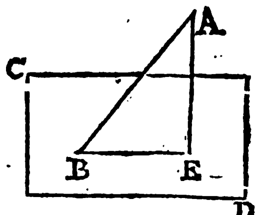
III. A right line AB is perpendicular to a Plane CD, when it makes right angles ABD, ABE, ABF, with all the right lines BD, BE, BF, that touch it, and are drawn in the said Plane.



IV. A Plane AB, is perpendicular to a Plane CD, when the right lines FG, HK, drawn in one Plane AB to the line of common section of the two Planes EB, and making right angles therewith, do also make right angles with the other Plane CD.

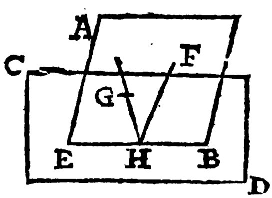
V. The

V. The inclination of a right line AB to a Plane CD, is when a perpendicular AE is drawn from A the higheſt point of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular



AE makes in the plane CD, to the end B of the ſaid line AB which is in the ſame plane, whereby the angle ABE which is contained under the inſiſting line AB, and the line drawn in the plane EB is acute.

VI. The inclination of a plane AB to a plane CD, is an acute angle FGH contained under the right lines FH, GH which being drawn in either of the planes AB, CD to the ſame point H



of the common ſection BE, make right angles FHB, GHB, with the common ſection BE.

VII Planes are ſaid to be inclined to other planes in the ſame manner, when the ſaid angles of inclination are equal one to another.

VIII. Parallel planes are thoſe which being prolonged never meet.

IX. Like ſolid figures are ſuch as are contained under like planes equal in number.

X. Equal and like ſolid figures are ſuch as are contained under like planes equal both in multitude and magnitude.

XI. A ſolid angle is the inclination of more than two right lines which touch one another, and are not in the ſame ſuperficies.

Or thus;

A ſolid angle is that which is contained under more than two plane angles not being in the ſame ſuperficies, but conſiſting all at one point.

XII. A Pyramide is a ſolid figure comprehended under divers planes ſet upon one plane (which is the baſe

baſe of the pyramide,) and gathered together to one point.

XIII A Priſme is a ſolid figure contained under planes, whereof the two oppoſite are equal, like, and parallel; but the others are parallelograms.

XIV. A Sphere is a ſolid figure made when the diameter of a ſemicircle abiding unmoved, the ſemicircle is turned round about, till it return to the ſame place from whence it began to be moved.

Coroll.

Hence, all the rays drawn from the center to the ſuperficies of a ſphere, are equal amongſt themſelves.

XV The Axis of a ſphere, is that fixed right line, about which the ſemicircle is moved.

XVI The Center of a ſphere, is the ſame point with the center of the ſemicircle

XVII. The Diameter of a ſphere, is a right line drawn thro' the center, and terminated on either ſide in the ſuperficies of the ſphere.

XVIII. A Cone is a figure made, when one ſide of a rectangled triangle (*viz.* one of thoſe that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it firſt moved. And if the fixed right line be equal to the other which containeth the right angle, then the Cone is a rectangled Cone: But if it be leſs, it is an obtuſe-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The Baſe of a Cone is the circle, which is deſcribed by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the ſides thereof, (namely, which contain the right angle) abiding fix'd, till the parallelogram be turned about to the ſame place, where it began to move.

XXII. The Axis of a Cylinder is that quieſcent right line, about which the parallelogram is turned.

XXIII. And the Baſes of a Cylinder are the circles which are deſcribed by the two oppoſite ſides in their motion.

XXIV. Like Cones and Cylinders, are thoſe both whoſe Axes and Diameters of their Baſes are proportional.

XXV. A

XXV. A Cube is a ſolid figure contained under ſix equal ſquares.

XXVI. A Tetraedron is a ſolid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a ſolid figure contained under eight equal and equilateral triangles

XXVIII. A Dodecaedron is a ſolid figure contained under twelve equal, equilateral, and equiangular Pentagones.

XXIX. An Icoſaedron is a ſolid figure contained under twenty equal and equilateral triangles

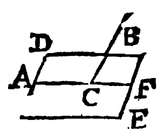
XXX. A Parallelepipedon is a ſolid figure contained under ſix quadrilateral figures, whereof thoſe which are oppoſite are parallel.

XXXI. A ſolid figure is ſaid to be inſcribed in a ſolid figure, when all the angles of the figure inſcribed are comprehended either within the angles, or in the ſides, or in the planes of the figure wherein it is inſcribed.

XXXII. Likewiſe a ſolid figure is then ſaid to be circumſcribed about a ſolid figure, when either the angles, or ſides, or planes of the circumſcribed figure touch all the angles of the figure which it contains.

PROPOSITION. I.

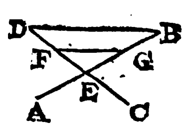
One part AC *of a right line cannot be in a plane ſuperficies, and another part of it* CB *above the ſame.*



Produce AC in the plane directly to F. If you conceive CB to be drawn ſtrait from AC, then two right lines AB, AF, have one common ſegment AC. a *Which is impoſſible.* a 10. *ax.* 1.

PROP. II.

If two right lines AB, CD, *cut one another, they are in the ſame plane: And every triangle* DEB *is in one and the ſame plane.*

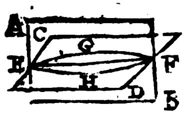


For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right line ED is in a plane, and the other part elevated upwards. a a 10. *ax.* 1.

Which

Which is absurd. Therefore the triangle EDB is in one and the same plane; and so also are the right lines ED, EB; *a* wherefore the whole lines AB, DC, are in one plane. *Which was to be dem.*

PROP. III.

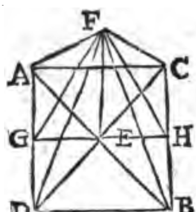


If two planes AB, CD, *cut one the other, their common section* EF *is a right line.*

If EF the common section be not a right line, *a* then in the plane AB draw the right line EGF. *a* and in the plane CD the right line EHF. therefore two right lines EGF. EHF include a superficies. b *Which is absurd.*

a 1. post. 1.
b 14. ax. 1.

PROP. IV.



If at E *the common section of two right lines* AB, CD, *a right line* EF *stands at right angles to them, it shall also be at right angles to the plane* ACBD *drawn thro' the said lines.*

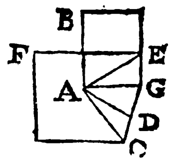
Take EA, EC, EB, ED, equal one to the other, and join the right lines AC, CB, BD, AD. draw any right line GH thro' E, and join FA, FC, FD, FB, FG, FH. Because AE is *a* = EB, and DE *a* = EC, and the angle AED *b* = CEB, *c* therefore AD is = CB, *c* likewise AC = DB. *d* therefore AD is parallel to CB, *d* and AC to BD. *e* wherefore the angle GAE = EBH, and the angle AGE = EHB. But also AE *f* = EB. *g* therefore GE = EH, *g* and AG = BH. whence by reason of the right angles, by the hyp. and so equal, at E, *h* the bases FA, FC, FB, FD, are equal. Therefore the triangles ADF, FBC, are equilateral one to another, *k* and thence the angle DAF = BCF. Therefore in the triangles AGF, FBH, the sides FG, FH *l* are equal; and so by consequence the triangles FEG and FEH are mutually equilateral. *m* therefore the angles FEG, FEH are equal, and *n* so right angles. In like manner, FE makes right angles with all the lines drawn thro' E in the plane ADBC, *o* and is therefore perpendicular to the said plane.

a constr.
b 15. 1.
c 4. 1.
d sch. 34. 1.
e 29. 1.
f constr.
g 26. 1.
h 4. 1.
k 8. 1.
l 4. 1.
m 8. 1.
n 10. def. 1.
o 3. def. 11.

PROP.

PROP. V.

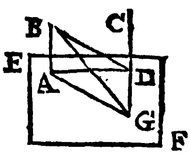
If a right line AB *be erected perpendicular to three right lines* AC, AD, AE, *touching one the other at the common section, those three lines are in the same plane*



For AC, AD, *a* are in one plane FC; *a* and AD, AE, are in one plane a 2. 11.
BE, which if you conceive to be several planes, then let their intersection *b* be the right line AG; there- b 3. 11.
fore because BA by the Hyp. is perpendicular to the right lines AC,AD. *c* and so to the plane FC, *d* it is also c 4. 11.
perpendicular to the right line AG. therefore (since *a* d 3. *def.* 11.
that AB is in the same plane with AG, AE) the angles BAG, BAE, are right angles, and consequently equal, the part and the whole. *Which is absurd.*

PROP. VI.

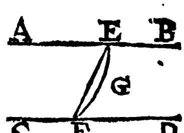
If two right lines AB, DC, *be erected perpendicular to one and the same plane* EF, *those right lines* AB, DC, *are parallel one to the other.*



Draw AD, whereunto let DG = AB be perpendicular in the plane EF, and join BD, BG, AG.
Because in the triangles BAD, ADG, the angles DAB, ADG *a* are right angles, and AB *b* = DG, and AD is a *hyp.*
common, *c* therefore BD is = AG. whence in the tri- b *constr.*
angles AGB, BGD, equilateral one to the other, the c 4. 1.
angle BAG is *d* = BDG; of which since BAG is a d 8. 1.
right angle, BDG shall be so also, but the angle GDC is supposed right, therefore the right line GD is perpendicular to the three lines DA, DB, CD. *e* which are e 5. 1.
therefore in the same plane *f* wherein AB is. Where- f 2. 11.
fore since AB and CD are in the same plane, and the internal angles BAD, CDA, are right angles, *g* AB g 28. 1.
and CD shall be parallels. *Which was to be dem.*

PROP. VII.

If there are two parallel right lines AB, CD, *and any points* E, F, *be taken in both of them, the line* EF *which is joined at these points, is in the same plane with the parallels* AB, CD.

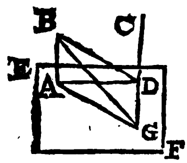


Let

Let the plane in which AB, CD, are, be cut by another plane at the points E, F. then if EF is not in the plane ABCD, it shall not be the common section. Therefore let EGF be the common section; which *a* then is a right line, therefore two right lines EF, EGF, include a superficies. b *Which is absurd.*

a 3. 11.
b 14. *ax.* 1.

PROP. VIII.

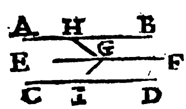


If there are two parallel right lines AB, CD, *whereof one* AB, *is perpendicular to a plane* EF. *then the other* CD *shall be perpendicular to the same plane* EF.

The preparation and demonstration of the sixth of this Book being transferr'd hither; the angles GDA, and GDB are right angles: *a* Therefore GD is perpendicular to the plane, wherein are AD, DB (*b* in which also AB, CD, are.) *c* therefore GD is perpendicular to CD. but the angle CDA is also *d* a right angle, *e* therefore CD is perpendicular to the plane EF. *Which was to be demonstrated.*

a 4. 11.
b 7. 11.
c 3. *def.* 11.
d 29. 1.
e 4. 11.

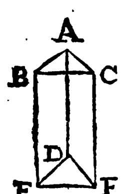
PROP. IX.



Right lines (AB, CD) *which are parallel to the same right line* EF, *but not in the same plane with it, are also parallel one to the other.*

In the plane of the parallels AB, EF, draw HG perpendicular to EF; also in the plane of the parallels EF, CD, draw IG perpendicular to EF. *a* therefore EG is perpendicular to the plane wherein HG, GI are; and AH, CI are perpendicular to the same plane, *c* therefore AH and CI are parallels. *Which was to be dem.*

PROP. X.



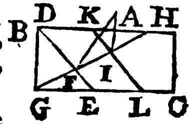
If two right lines AB, AC, *touching one another be parallel to two other right lines* ED, DF, *touching one another, and not being in the same plane, those right lines contain equal angles,* BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AD, BC, EF, BE, CF. Since AB, DE, *a* are parallels and equal, *b* also BE, AD, are parallels and equal. In like manner

a *hyp* and *constr*
b 33. 1.

manner CF, AD, are parallel and equal; *c* therefore c 2. *ax.* 1.
alſo BE, FC, are parallel and equal. *d* Therefore BC, *and* 9. 11.
EF are equal. Wherefore ſince the triangles BAC, d 33. 1.
EDF, are of equal ſides one to the other, the angles e 8. 1.
BAC, EDF *e* ſhall be equal. *Which was to be dem.*

PROP. XI.



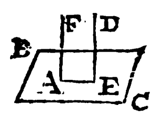
From a point given on high A *to draw a right line* AI *perpendicular to a plane below* BC.

In the plane BC draw any line DE; to which from the point A *a* draw the perpendicular AF, to the ſame DE through F in the plane BC *b* draw the perpendicular FH, then to FH *a* draw the perpendicular AI, this ſhall be perpendicular to the plane BC.

a 12. 1.
b 11. 1.
c 31. 1.
d *conſtr.*
e 4. 11.
f 8. 11.
g 3. *def* 11.
h *conſtr.*
l 4. 11.

For thro' I *c* let KIL be drawn parallel to DE. Becauſe DE *d* is perpendicular to AF, and FH, *e* therefore DE ſhall be perpendicular to the plane IFA. and ſo alſo KL *f* is perpendicular to the ſame plane, *g* therefore the angle KIA is a right angle, but the angle AIF is alſo *h* a right angle, *l* therefore AI is perpendicular to the plane BC. *Which was to be done.*

PROP. XII.



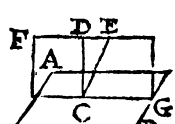
In a plane given BC, *at a point given therein* A, *to erect a perpendicular line* AF.

From ſome point D without the plane, *a* draw DE perpendicular to the ſaid plane BC and joining the points A, E, by a line AE, *b* draw AF parallel to DE. *c* it is apparent that AF is perpendicular to the plane BC. *Which was to be done*

a 11. 11.
b 31. 1.
c 8. 11.

This and the preceding problem are practically performed by applying two Squares to the point given; as appears by 4. 11.

PROP. XIII.



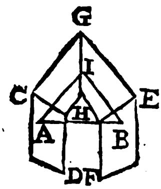
At a point given C *in a plane given* AB, *two right lines* CD, CE, *cannot be erected perpendicular to the ſaid given plane on the ſame ſide.*

For both CD, and CE *a* ſhould then be perpendicular to the plane AB, and conſequently a 6. 11.

 paral-

parallels; which is repugnant to the definition of parallel lines.

PROP. XIV.

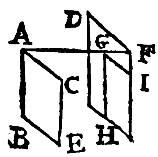


Planes CD, FE, *to which the same right line* AB *is perpendicular, are parallel.*

If you deny this; then let the planes CD, EF meet, so that their common section be the right line GH, in which take any point I, draw to it the right lines IA, IB, in the said planes. whereby in the triangle IAB, two angles
a *hyp and* 3. *def.* 11. IAB, IBA *a* are right angles. b *Which is absurd.*

b 17. 1.

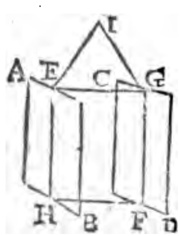
PROP. XV.



If two right lines AB, AC, *touching one the other, are parallel to two other right lines* DE, DF, *touching one the other, and not being in the same plane with them, the planes* BAC, EDF, *drawn by those right lines are parallel one to the other.*

a 11. 11. From A *a* draw AG perpendicular to the plane EF.
b 31. 1. *b* and let GH, GI be parallel to DE, DF. *c* these also
c 9. 11. shall be parallel to AB, AC. Therefore since the an-
d 3 *def.* 11. gles IGA, HGA, *d* are right angles, also CAG, BAG,
e 29. 1. *e* shall be right angles. *f* therefore GA is perpendicu-
f 4. 11. lar to the plane BC; but the same is perpendicular to
g *constr.* the plane EF, *h* therefore the planes BC, EF, are pa-
h 14. 11. rallel. *Which was to be dem.*

PROP. XVI.

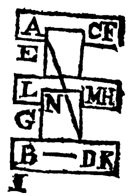


If two parallel planes AB, CD, *are cut by some other plane* HEIGF, *their common sections* EH, GF *are parallel one to the other.*

For if they are said to be not parallel, then, since they are in the same cutting plane, they must meet some where, suppose in I, wherefore since the whole lines HEI,

HEI, FGI *a* are in the planes AB, CD, produced, the planes also shall meet. *contrary to the Hyp.* a 1. 11.

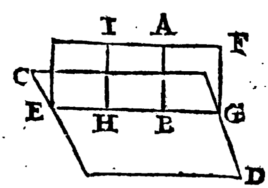
PROP. XVII.



If two right lines ALB, CMD, *are cut by parallel planes* EF, GH, IK; *they shall be cut proportionally*, (AL. LB :: CM. MD.)

Let the right lines AC, BD, be drawn in the planes EF, IK; as also AD meeting the plane GH in the point N. and join NL, NM, the planes of the triangles ADC, ADB, make the sections BD, LN, and AC, NM, *a* parallels. Therefore AL. LB :: AN. ND *b* :: CM. MD. *Which was to be dem.* a 16. 11. b 2. 6.

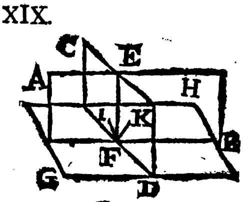
PROP. XVIII.



If a right line AB *be perpendicular to some plane* CD, *all the planes* EF *passing thro' that right line* AB *shall be perpendicular to the same plane* CD.

Let there be some plane BF drawn thro' AB, making the section EG with the plane CD; from some point whereof H, *a* draw HI parallel to AB in the plane EF; *b* then shall HI be perpendicular to the plane CD, and so likewise any other lines, that are perpendicular to EG. *c* therefore the plane EF is perpendicular to the plane CD; and for the same reason any other planes drawn thro' AB shall be perpendicular to CD. *Which was to be dem.* a 31. 1. b 8. 11. c 4. *def.* 11.

PROP. XIX.



If two planes AB, CD, *cutting one the other, are perpendicular to some plane* GH, *their line of common section* EF *shall be perpendicular to the same plane* (GH)

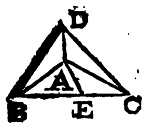
Because the planes AB, CD,

 are

are taken perpendicular to the plane GH, it appears by 4. def. 11. that from the point F there may be drawn in both planes AB, CD, a perpendicular to the plane GH, which shall be *a* one and the same line, and therefore the common section of the said planes. *Which was to be demonstrated.*

a 13. 11.

PROP. XX.



If a solid angle ABCD *be contained under three plane angles*, BAD, DAC, BAC, *any two of them howsoever taken are greater than the third.*

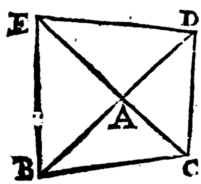
If the three angles are equal, the assertion is evident; if unequal, then let the greatest be BAC; from whence *a* take away BAE = BAD, and make AD = AE; and also draw BEC, BD, DC.

a 23. 1.

Because the side BA is common, and AD *b* = AE; and the angle BAE *b* = BAD, *c* thence is BE = BD. but BD + DC is *d* ⊏ BC. *e* therefore DC ⊏ EC. Wherefore since AD *b* = AE, and the side AC is common, and DC ⊏ EC. *f* the angle CAD shall be ⊏ EAC, *g* therefore the angle BAD + CAD. ⊏ BAC. *Which was to be dem.*

b *constr.*
c 4. 1.
d 20 1.
e 5. *ax.* 1.
f 25. 1.
g 4 *ax.* 1.

PROP. XXI.



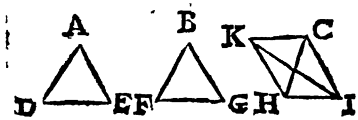
Every solid angle A *is contained under less angles than four plane right angles.*

For let a plane any-wise cutting the sides of the solid angle A make a many-sided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X; and I term the sum of the angles at the bases of the triangls Y. whereof X + 4 right angles *a* = Y + A. but because that (of all the angles at *B*) *b* the angle ABE + ABC is ⊏ CBE, and the same is true also of the angles at C, at D, and at E, *c* it is manifest that Y is ⊏ X, and consequently A shall be ⊐ 4 right angles. *Which was to be dem.*

a 32 1 & *sch.* 31 1.
b 20. 11.
c 5. *ax.* 1.

PROP.

PROP. XXII.

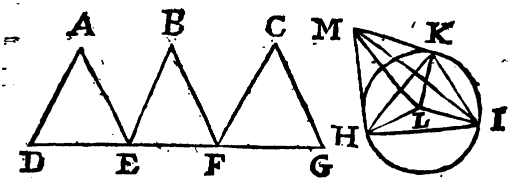


If there are three plane angles A, B, HCI, *whereof two howsoever taken are greater than the third, and the right lines which contain them are equal* AD, AE, FB, &c. *then of the right lines* DE, FG, HI, *connecting those equal right lines together, it is possible to make a triangle.*

A triangle may be *a* made of them, if any two be greater than the third; but they are so. For *b* make the angle HCK = B, and CK = CH, and draw HK, IK. *c* thence KH = FG and because the angle KCI *d* ⊏ A. *e* therefore KI ⊏ DE, but KI *f* ⊐ HI + KH (FG.) therefore DE ⊐ HI + FG. By the like argument any other two may be proved greater than the third; and consequently *a* it is possible to make a triangle of them. *Which was to be dem.*

a 22. 1.
b 23. 1.
c 4. 1.
d *hyp.*
e 24. 1.
f 20. 1.

PROP. XXIII.



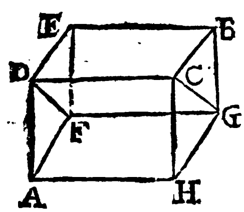
To make a solid angle MHIK *of three plane angles* A, B, C, *whereof two howsoever taken are greater than the third.* * *But it is necessary that those three angles be less than four right angles.* * 21. 11.

Make AD, AE, BE, BF, CF, CG, equal one to the other; and of the subtended lines DE, EF, FG (that is, of the equal lines HI, IK, KH) *a* make the triangle HKI; about which *b* describe the circle LHKI. * But because AD is ⊏ HL, *c* let ADq be = HLq + LMq. *d* and let LM be perpendicular to the plane of the circle HKI. and draw HM, KM, IM. where-

a 22. 11. *and* 21. 1.
b 5. 4.
* *See Clavius.*
c *sch.* 47. 1.
d 12. 11.

e 3. *def.* 11. wherefore since the angle HLM *e* is a right angle, *f*
f 47. 1. thence is MHq = HLq + LMq *g* = ADq. therefore
g *constr.* MH = AD. By the same way of reasoning MK, MI,
h *constr.* AD (that is AE, EB, &c.) are equal; therefore since
k 8. 1. HM = AD, and MI = AE, and DE *h* = HI, *k* the
angle A shall be = HMI, *k* as likewise the angle IMK
= B, *k* and the angle HMK = C, wherefore a solid
angle is made at M of the three given plane angles.
Which was to be done. AD is assumed to be $\sqsubset$ HL.
But this is manifest. For if AD be = or $\sqsupset$ HL, then
l *constr.* & is the angle A *l* = *m* or $\sqsubset$ HLI. In like manner
8. 1. shall B be equal or $\sqsubset$ HLK, and C = or $\sqsubset$ KLI,
m 21. 1. wherefore A + B + C * shall either equal or exceed
*4. *cor.* 13. 1 four right angles. *contrary to the Hyp.* therefore rather
let AD be $\sqsubset$ HL. *Which was to be dem.*

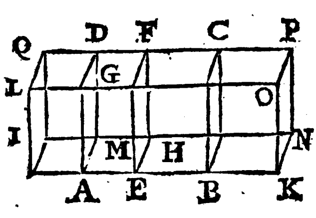
PROP. XXIV.



If a solid AB be contained under parallel planes, the opposite planes thereof (AG, BD, &c) are like and equal parallelograms.

a 16. 11. The plane AC cutting the parallel planes AG, DB, *a* makes the sections AH, DC, parallels, and for the same reason AD, HC are parallels. Therefore ADCH is a pgr. By the like argument the other planes of the pa-
b 35. *def.* 1. rallelepipedon are *b* pgrs. wherefore since AF is paral-
c 10. 11. lel to HG, and AD to HC, *c* the angle FAD shall be
d 34. 1. = GHC, therefore because AF *d* = HG, and AD *d* =
e 7. 5. HC, and so AF. AD :: HG. HC, the triangles FAD,
g 6. 6. GHC *g* are like and *h* equal; and consequently the pgrs.
h 4. 1. AE, HB are like and *k* equal, and the same may be
k 6 *ax.* 1. shewn of the rest of the opposite planes, therefore, &c.

PROP. XXV.



If a solid Parallelepipedon ABCD *be cut by a plane* EF *parallel to the opposite planes* AD, BC; *then as the base* AH *is to the base* BH, *so shall solid* AHD *be to solid* BHC.

Conceive

Conceive the parallelepipedon ABCD to be extended on either side, and take AI = AE, and BK = EB, and put the planes IQ, KP, parallel to the planes AD, BC; then the pgrs. IM, AH, and *a* DL, DG, *b* and IQ, AD, EF, &c. are *a* like and equal, *c* wherefore the Parallelepipedon AQ is = AF; and for the same reason the Parallelepipedon BP = BF. therefore the solids IF, EP are as multiple of the solids AF, EC, as the bases IH, KH, are of the bases AH, BH. And if the base IH be ⊏, =, ⊐ KH, *d* likewise shall the solid IE be ⊏, =, ⊐ EP. *e* consequently AH. BH :: AF. EC. *Which was to be dem.*

a 36. 1. *and* 1 *def.* 6. b 24. 11. c 10 *def.* 11. d 24. 11. *and* 9 *def.* 11. e 6. *def.* 5.

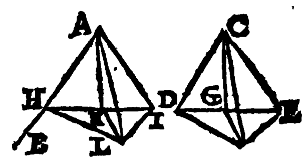
The same may be accomodated to all sorts of prismes, whence

Coroll.

If any prisme whatsoever be cut by a plane parallel to the opposite planes, the section shall be a figure equal and like to the opposite planes.

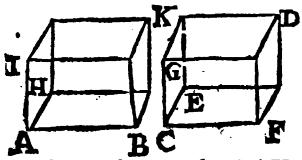
PROP. XXVI.

Upon a right line given AB, *and at a point given in it* A, *to make a solid angle* AHIL *equal to a solid angle given* CDEF.



From some point F *a* draw FG perpendicular to the plane DCE, and draw the right lines DF, FE, EG, GD, CG. Make AH = CD, and the angle HAI = DCE, and AI = CE; and in the plane HAI make the angle HAK = DCG, and AK = CG, then erect KL perpendicular to the plane HAI, and let KL be = GF, and draw AL: Then AHIL shall be a solid angle equal to that given CDEF. For the construction of this does wholly resemble the framing of that, as will easily appear to any who examine it. a 11. 11.

PROP. XXVII.

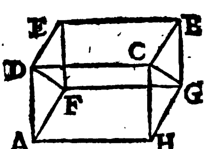


Upon a right line given AB *to describe a parallelepipedon* AK, *like, and in like manner situate, with a solid parallelepipedon given* CD.

Of the plane angles, BAH, HAI, BAI, which are e-
a 26. 11. qual to FCE, ECG, FCG, *a* make the solid angle A e-
b 12. 6. qual to the solid angle C. also *b* make FC. CE :: BA.
c 22. 5. AH. *b* and CE. CG :: AH. AI (*c* whence by equality FC. CG :: BA. AI) and finish the parallelepipedon AK, which shall be like to that which is given.

For by the construction, the Pgr. *d* BH is like to FE,
d 1. def. 6. and *d* HI to EG, and *d* BI to FG, and *e* so the opposites
e 24. 11. of these to the opposites of them: Therefore the six planes of the solid AK are like to the six planes of the
f 9. def. 11. solid CD, *f* and consequently AK, CD, are like solids. *Which was to be dem.*

PROP. XXVIII.

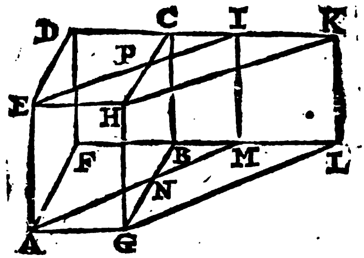


If a solid parallelepipedon AB *be cut by a plane* FGCD *drawn thro' the diagonal lines* DF, CG, *of the opposite planes* AE, HB, *that solid* AB *shall be equally bisected by the plane* FGCD.

a 24. 1. For because DC, FG, are *a* equal and parallels, *b* the
b 34. 1. plane FGCD is a Pgr and because *a* the Pgrs. AE HB, are equal and like, *b* also the triangles AFD, HGC, CGB, DFE are equal and like. But the Pgrs. AC. AG, are equal and like to FB and FD, therefore all the planes of the prisme FGCDAH are equal and like to
c 9. def 11. all the planes of the prisme, FGCDEB, and *c* consequently this prisme is equal to that. *Which was to be demonstrated.*

PROP.

PROP. XXIX.



Solid parallelepipedons AGHEFBCD, AGHEMLKI, *being constituted upon the same base* AGHE, *and* * *in the same height, whose insisting lines* AF, AM, *are placed in the same right lines* AG, FL, *are equal one to the other.*

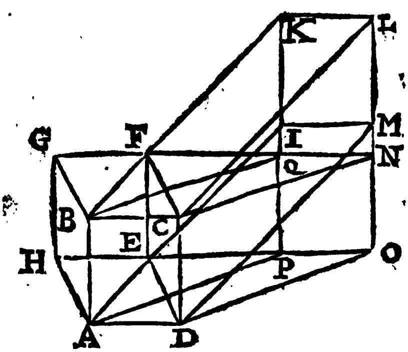
For *a* if from the equal prismes AFMEDI, GBLHCK, the common prisme NBMPCI be taken away, and the solid AGNEHP be added, the Parallelepipedon AGHEFBCD shall be = AGHEMLKI. *Which was to be demonstrated.*

* *i. e. between the parallel planes* AGHE, FLKD. & *so understand it in the fol.*

a 10. *def* 11 & 35. 1.

b 3. *and* 2*1* *ax.* 1.

PROP. XXX.

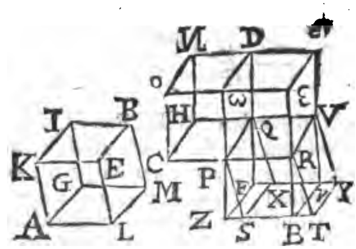


Solid parallelepipedons ADBCHEFG, ADCBIMLK *being constituted upon the same base* ADBC, *and in the same height, whose insisting lines* AH, AI *are not placed in the same right lines, are equal one to the other.*

For produce the right lines HEO, GFN, and LMO, KIP;

a 34. 1. KIP; and draw AP, DO. BQ. CN. *a* then shall DC, AB, HG, EF, PQ, ON be as well equal and parallel one to
b 29. 11. the other as AD, HE, GF, BC, KL, IM, QN. PO. *b* wherefore the parallelepipedon ADCBPONQ shall be
c 1. ax. 1 equal to either parallelepipedon ADCBHEFG, ADCBIMLK; and *c* consequently these two are equal one to the other. *Which was to be dem.*

PROP. XXXI.



Solid parallelepipedons, ALEKGMBI, CPωOHQDN, *being constituded upon equal bases* ALEK, CPωO, *and* in the same height are equal, one to the other.*

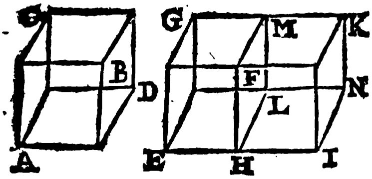
**by height understand the perpendicular drawn from the plane of the base to the opposite plane.*

First, let the parallelepipedons AB, CD, have the sides perpendicular to the bases, and at the side CP being produced, *a* make the Pgr. PRTS equal and like to the pgr. KELA. *b* and so the parallelepipedon PRTSQVYX equal and like to the parallelepipedon AB. Produce OωE, NDδ, ωPZ, DQF, ERB, δVγ, TSZ, YXF; and draw Eδ, Bγ, ZF.

a 18. 6 The planes OεδN, CRVH, ZTYF, *c* are parallels
b 27. 11 & 10. def. 11. one to the other; *d* and the Pgrs. ALEK, CPωO. PRTS, PRBZ are equal. Therefore since the paralle-
c 30. def. 11. pipedon CD. PVδω *e* :: Pgr. Cω (PRBZ) Rω :: paral-
d hyp. and 35. 1. lelepipedon PRBZQVγF. PVδω; the parallelepipedon CD *f* shall be = PRBZQVγF *g* = PRVQSTYX *h* =
e 25. 11. AB. *Which was to be dem.*
f 9. 5. But if the parallelepipedons AB, CD, have sides ob-
g 29. 11. lique to the base, then on the same bases and in the
h constr. same height place parallelepipedons whose sides are per-
k 29. 11. pendicular to the base. *k* They shall be equal to one ano-
m 1. ax. 1. ther, and to those that are oblique, *m* whence also the oblique parallelepipedons AB, CD are equal. *Which was to be demonstrated.*

PROP.

PROP. XXXII.

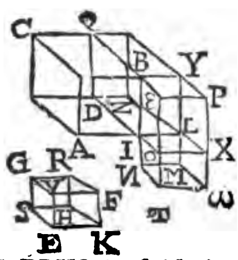


Solid parallelepipedons ABCD, EFGL, *of the same height, are one to the other, as their bases,* AB, EF.

Produce EHI, *a* and make the pgr. FI = AB, and a 45. 1.
b compleat the parallelepipedon FINM. It is clear that b 27. 11.
the parallelepipedon FINM. (*c* ABCD) EFGL *d* :: FI c 31. 11.
(AB) EF. *Which was to be dem.* d 25. 11.

PROP. XXXIII.

Like solid parallelepipedons, ABCD, EFGH, *are to one another in triplicate ratio of their homologous sides* AI, EK.



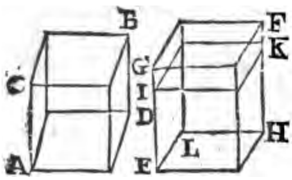
Produce the right lines AIL, DIO, BIN, and *a* make IL, IO, IN, equal to EK, KH, KF, *b* and so the parallelepipedon IXMT equal and like to the parallelepipedon EF- a 3. 1. b 27. 11.
GH. *c* Let the parallelepps. IXPB, DLYQ be finished. *d* c 31. 1.
Then shall be AL. IL (EK) :: DI. IO (HK) :: BI. IN. d *hyp.*
(KF) *e* that is the pgr. AD. DL :: DL. IX :: BO. IT. e 1. 6.
f i.e the parallepp. ABCD. DLQY :: DLQY. IXBP f 32. 11.
:: IXBP. IXMT. (*g* EFGH.) *h* therefore the proporti- g *constr.*
on of ABCD to EFGH is triplicate of the proportion h 10. *def.* 5.
of ABCD to DLQY, *k* or of AI to EK. *Which was to* k 1. 6.
be demonstrated.

Coroll.

Hence it appears that if four right lines be continually proportional, as the first is to the fourth, so is a parallelepipedon described on the first to a parallelepipedon described on the second, being like and in like manner described.

PROP.

PROP. XXXIV.



In equal ſolid parallelepipedons ADCB, EHGF, *the baſes and altitudes are reciprocal* (AD. EH :: EG. AC) *And ſolid parallelepipedons*, ADCB, EHGF, *whoſe baſes and altitudes are reciprocal, are equal.*

First, let the ſides CA, GE be perpendicular to the baſes; then if the altitudes of the ſolids are equal, the baſes alſo ſhall be equal, and the thing is clear. But if
a 3. 1. the altitudes are unequal, from the greater EG *a* take
b 31. 1. EI = AC, and at I *b* draw the plane IK parallel to
c 32. 11. the baſe EH. then
d 17. 5. 1. *Hyp.* AD. EH *c* :: parallepp. ADCB. EHIK *d* ::
e 1. 6. parallepp. EHGF. EHIK *c* :: GL. IL *e* :: GE. IE (*f*
f *conſtr.* AC.) *g* it is plain therefore that AD. EH :: GE. AC
g 11. 5. *Which was to be dem.*
h 32. 11. 2 *Hyp.* ADCB. EHIK *h* :: AD. EH *k* :: EG. EI *l* ::
k *hyp.* GL. IL *m* :: parallepp. EHGF. EHIK, *n* wherefore the
l 1. 6. parallelepipedon ADCB = EHGF. *Which was to be*
m 32. 11. *demonſtrated.*
n 9 5. Moreover, let the ſides be oblique to the baſes and erect right parallelepipedons upon the ſame baſes in the ſame altitude; the oblique parallepps. ſhall be equal to them. Wherefore ſince by the firſt part, the baſes and altitudes of thoſe are reciprocal, the baſes and altitudes of theſe alſo ſhall be reciprocal. *Which was to be dem.*

Coroll.

All that hath been dem. of parallepps. in the 29, 30, 31, 32, 33, 34. Prop. *does alſo agree to triangular priſmes, which are half parallepps. as appears by Prop.* 28. Therefore,

1. Triangular priſmes are of equal height with their baſes.

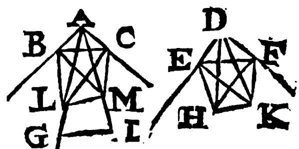
2. If they have the ſame or equal baſes and the ſame altitude, they are equal.

3. If they are like, their proportion is triplicate of that of their homologous ſides.

4. If they are equal, their baſes and altitudes are reciprocal; and if their baſes and altitudes are reciprocal, they are alſo equal.

PROP.

PROP. XXXV.



If there are two plane angles BAC, EDF, *equal, and from the point of those angles two right lines* AG, DH, *be elevated on high, containing equal angles with the lines first given, each to his correspondent angle (the angle* GAB = HDE, *and* GAC = HDF) *and if in those elevated lines* AG, DH, *some points be taken*, G, H; *and from these points perpendicular lines* GI, HK, *drawn to the planes* BAC, EDF, *in which the angles first given are, and right lines* AI, DK, *be drawn to the angles first given from the points* I, K, *which are made by the perpendiculars in the planes; those right lines with the elevated lines* AG, DH *shall contain equal angles* GAM, HDK.

Make DH, AL, equal; and GI, LM parallels, and MC to AC, MB to AB, KF to DF, KE to DE perpendicular; and draw the right lines BC, LB, LC, and EF, HF, HE; *a* and LM is perpendicular to the plane BAC; a 8. 11.
b wherefore the angles LMC, LMA, LMB; and for the same reason the angles HKF, HKD, HKE are right angles. Therefore ALq *c* = LMq + AMq *c* = LMq + CMq + ACq *c* = LCq + ACq, *d* therefore the Angle ACL is a right angle. Again ALq *e* = LMq + MAq *e* = LMq + BMq + BAq *e* = BLq + BAq. *d* therefore the angle ABL is also a right angle. By the like inference the angles DFH, DEH are right angles; *f* therefore AB = DE, *f* and BL = EH, *f* and AC = DF, and CL = FH, *g* wherefore also BC = EF; *g* and the angle ABC = DEF, *g* and the angle ACB = DFE. *h* whence the other right angles CBM, BCM, are equal to the other FEK, EFK. *k* therefore CM = FK, *l* and so also AM = DK. therefore if from LAq *m* = HDq be taken away AMq = DKq, *n* there remains LMq = HKq, wherefore the triangles LAM, HDK are equilateral one to the other; *o* therefore the angle LAM = HDK. *Which was to be dem.*

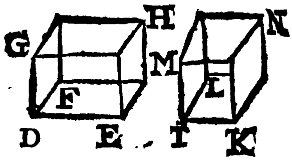
b 3. *def.* 11.
c 47. 1.
d 48. 1.
e 47. 1.
f 26. 1.
g 4. 1.
h 3. *ax.* 1.
k 26. 1.
l 47. 1.
m *constr.*
n 47. 1. & 3. *ax.*
o 8. 1.

Coroll.

Therefore, if there be two plane angles equal, from whose points equal right lines are elevated on high containing

taining equal angles with the lines first given, each to each; perpendiculars drawn from the extreme points of those elevated lines to the planes of the angles first given, are equal one to the other, *viz.* LM = HK.

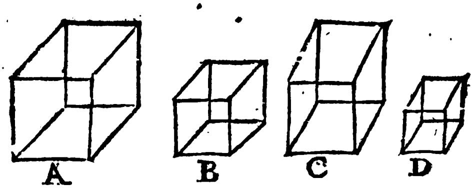
PROP. XXXVI.



If there are three right lines DE, DG, DF *proportional, the solid parallelpp.* DH. *made of them, is equal to the solid parallelpp.* IN *made of the middle line* DG (IL) *which is also equilateral, and equiangular to the said parallelepipedon* DH.

a *hyp.* Because DE. IK *a* :: IL. DF, *b* the pgr. LK shall be
b 14. 6. = FE, and by reason of the equality of the plane angles at D and I, and of the lines GD, IM, also the altitudes of the parallelepps. are equal by the preceding Coroll.
c 31. 11. *c* therefore the parallelepps. are equal one to the other. *Which was to be dem.*

PROP. XXXVII.

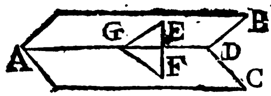


If there are four right lines A, B, C, D, *proportional, the solid parallelepps.* A, B, C, D *being like, and in like sort described from them, shall be proportional. And if the solid parallepps. being like and in like sort described, be proportional* (A. B :: C. D.) *then those right lines* A, B, C, D, *shall be proportional.*

a 33. 11. For the proportions of the parallelepps. *a* are tripli-
b *sch.* 23 5. cate of those lines; therefore if A. B :: C. D, *b* then shall the parallelpp. A. parallelpp. B :: parallelpp. C. parallelpp. D. and so also contrarily.

PROP.

PROP. XXXVIII.

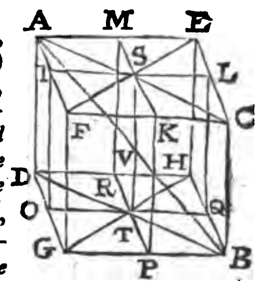


If a plane AB *be perpendicular to a plane* AC, *and a perpendicular line* EF *be drawn from a point* E *in one of the planes* (AB) *to the other plane* AC, *that perpendicular* EF *shall fall upon the common section of the planes* AD.

If it be possible, let F fall without the intersection AD. and in the plane AC *a* draw FG perpendicular to AD, and join EG. The angle FGE *b* is a right angle, and EFG is supposed to be such also; therefore two right angles are in the triangle EFG. *c Which is absurd.*

a 12. 1.
b 4 *and* 3. *def.* 11.
c 17. 1.

PROP. XXXIX.



If the sides (AE, FC, AF, EC, *and* DH, GB, DG, HB) *of the opposite planes* AC, DB, *of a solid parallelpp.* AB, *be divided into two equal parts, and planes* ILQO, PKMR, *be drawn thro' their sections, the common section of the planes* ST, *and the diameter of the solid parallelpp.* AB *shall divide one the other into two equal parts.*

Draw the right lines SA, SC, TD, TB. Because *a* the sides DO, OT are equal to the sides BQ, QT, *b* and the alternate angles TOD, TQB equal, also *c* the bases DT, TB, and the angles DTO, BTQ are equal, *d* therefore DTB is a right line. and so in like manner is ASC. Moreover *e* as well AD is parallel and equal to FG *e* as FG to CB, and *f* thence AD is parallel and equal to CB; *g* and conseqently AC to DB *h* wherefore AB and ST are in the same plane ABCD. Therefore since the vertical angles AVS, BVT, and the alternate angles ASV, BTV are equal; *k* and AS = BT; therefore shall AV be = BV, *l* and SV = VT. *Which was to be dem.*

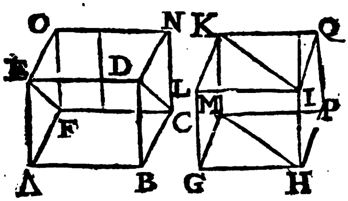
a 34. 1.
b 29. 1.
c 4. 1.
d *sch.* 15. 1.
e 34. 1.
f 9. 11. *and* 1 *ax.*
g 33. 1.
h 7. 11.
k 7. *ax.* 1.
l 26. 1.

Coroll.

Hence in every parallelepipedon. all the diameters bisect one another in one point, V.

PROP.

PROP. XL.



If two prismes ABCFED, GHMLIK, *be of equal altitude, whereof one hath its base* ABCF *a parallelogram, and the other* GHM *a triangle; and if the parallelogram* ABCF *be double to the triangle* GHM; *those prismes* ABCFED, GHMLIK *are equal.*

a 31. 11. For if the parallelepps. AN, GQ, be compleated, *a*
b 34. 1 *and* they shall be equal, because of the equality *b* of the ba-
7. *ax.* ses AC, GP, and *c* of the altitudes, *d* therefore also the
c *hyp.* prismes, *e* the halfs theereof shall be qual. *Which was*
d 28. 11. *to be dem.*
e 7. *ax.* 1.

Schol.

From the preceding demonstrations, the demension of triangular prismes, and quadrangular, or parallelepps. is learnt; viz. *by multiplying the altitude into the base.* *Andr. Tacq*

As if the altitude be 10 foot, and the base 100 square foot (the base may be measured by *sch.* 35. 1. or by 41. 1) then multiply 100 by 10, and 1000 cubic foot shall be produced for the solidity of the prisme given.

For as a rectangle, so also is a right parallelepp. produced from the altitude multiplied into the base. Therefore every parallelepp. is produced from the altitude multiplied into the base, as appears by 31. of this Book.

Moreover, since the whole parallelepp. is produced from the altitude drawn into the base, the half thereof (that is, a triangular prisme) shall be produced from the altitude drawn into half the base, namely the triangle.

An Advertisement

Obs. *That of those letters which denote a solid angle, the first is always at the point in which the angle is; but of those letters which denote a pyramide, the last is at the supreme point thereof.*

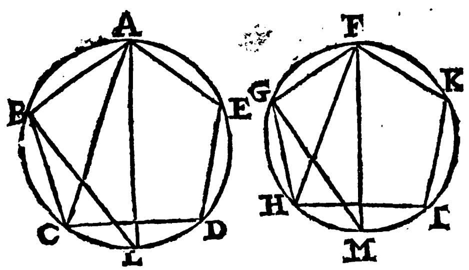
Ex. gr. the solid angle ABCD is at the point A; and the supreme point of the pyramide BCDA is at the point A. and the base is the triangle BCD.

The End of the eleventh Book.

THE

The Twelfth Book OF EUCLIDE'S ELEMENTS.

PROPOSITION I.



LIKE polygonous figures ABCDE, FGHIK *inscribed in circles* ABD, FGI, *are one to another, as the squares described on the diameters of the circles* AL, FM.

Draw AC, BL, FH, GM. Because *a* the angle ABC = FGH, *a* and AB. BC : : FG. GH *b* therefore shall the angle ACB (*c* ALB) be = FHG (*c* FMG.) but the angles ABL, FGM *d* are right and so equal; *e* therefore the triangles ABL, FGM are eqiangular, *f* wherefore AB. FG :: AL. FM. *g* therefore ABCDE. FGHIK :: ALq. FMq.

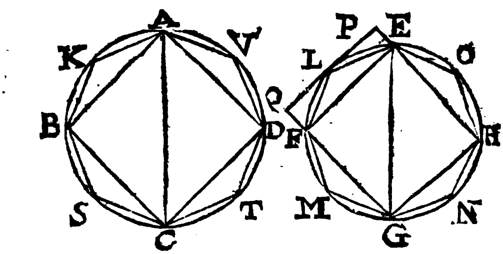
a 1. *def.* 6.
b 6. 6.
c 21. 3.
d 31. 3.
e 32. 3.
f *cor.* 4. 6.
g 22. 6.

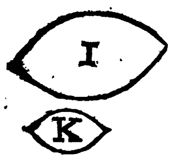
Coroll.

Hence (because AB. FG :: AL. FM :: BC. GH, &c.) the ambits of like polygonous figures inscribed in a circle are in *h* proportion as the diameters.

h 1. 12. *and* 12. 5.

PROP. II.





Circles ABT, EFN, *are in proportion one to another, as the squares of their diameters* AC, EG *are.*

Suppose ACq. EGq :: the circle ABT. I. I say then I is equal to the circle EFN.

For first, if it be possible, let I be less than the circle EFN, and let K be the excess or difference. Inscribe the square EFGH in the cir-
a *sch.* 7. 4. cle EFN, *a* it being the half of a circumscribed square,
b 30. 3. and so greater than the semicircle. *b* Divide equally in two the arches EF, FG, GH, HE, and at the points of the divisions join the right lines EL, LF, &c. thro' L
c *sch* 27. 3. draw the tangent PQ (*c* which is parallel to EF) and
d 41. 1. produce HEP, GFQ. then is the triangle ELF *d* the half of the pgr. EPQF, and so greater than the half of the segment ELF; and in like sort the rest of those triangles exceed the halfs of the rest of the segments. And if the arches EL, LF, FM, &c. be again bisected, and the right lines joined, the triangles will likewise exceed the half of the segments Wherefore if the square EFGH be taken from the circle EFN, and the triangles from the other segments, and this be done
e 1. 10. continually, at length *e* there will remain some magnitude less than K. Let us have gone so far, namely, to
f. *hyp. and* the segments EL, LF, FM, &c. taken together less
3. *ax.* than K. Therefore I (*f* the circle EFN — K) ⊐
g 30. 3 & the polyg ELFMGNHO (the circle EFN — the seg-
1. *post* 1. ment EL + LF, &c.) In the circle ABT *g* conceive
h 1. 12. a like polygon AKBSCTDV inscribed. therefore since
k *hyp.* AKBSCTDV. ELFMGNHO *h* :: ACq. EGq *k* :: the circle

circle ABT. I. and the polyg. AKBSCTDV *l* $\sqsupset$ the circle ABT. the polyg. ELFMGNHO *m* shall be $\sqsupset$ I. but before, I was $\sqsupset$ ELFMGNHO. which is repugnant. l 9. *ax.* 1. m 14. 5.

Again, if it be possible, let I be $\sqsubset$ the circle EFN. Therefore because ACq. EGq *n* :: the circle ABT I; and inversely I. the circle ABT :: EGq. ACq. suppose I. the circle ABT :: the circle EFN. K. *o* therefore the circle ABT $\sqsubset$ K. *p* and EGq. ACq :: the circle EFN. K. which was just now shewn to be repugnant. n *hyp.* o 14. 5. p 11. 5.

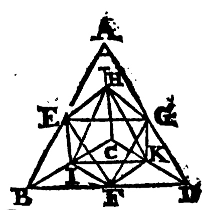
Therefore it must be concluded, that I is = to the circle EFN. *Which was to be dem.*

Coroll.

Hence it follows, that as a circle is to a circle, so is a polygon inscribed in the first to a like polygon inscribed in the second.

PROP. III.

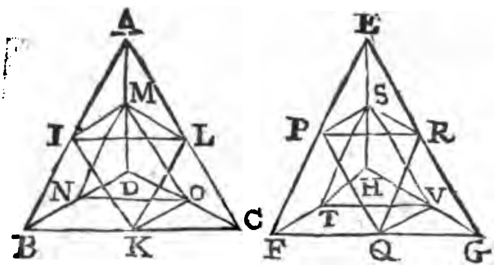
Every Pyramide ABDC *having a triangular base, may be divided into two pyramides* AEGH, HIKC, *equal, and like one to the other, having bases triangular, and like to the whole* ABDC; *and into two equal prismes,* BFGEIH, FGDIHK; *which two prismes are greater than the half of the whole pyramide* ABDC



Divide the sides of the pyramide into two parts at the points E, F, G, H, I, K, and join the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Because the sides of the pyramide are proportionally cut, *a* thence HI, AB; and GF, AB; and IF, DC; and HG, DC, &c. are parallels, and consequently HI, FG; and GH, FI are also parallels, therefore it is apparent that the triangles ABD, AEG, EBF, FDG, HIK, *b* are equiangular, and that the four last are *c* equal: In like manner the triangles ACB, AHE, EIB, HIC, FGK are equiangular; and the four last are equal one to the other. Also the triangles BFI, FDK, IKC, EGH; and lastly, the triangles AHG, GDK, HKC, EFI are like and equal. Moreover the triangles, HIK to ADB, EGH to BDC, and EFI to ADC, and FGK to ABC, *d* are parallel From whence a 2. 6. b 29. 1. c 26. 1. d 15. 11.

whence it evidently follows, first, that the pyramides
e 10. def. 11. AEGH, HIKC are equal, and e like to the whole ABDC, and to one another. Next, that the solids BFGEIH, FGDIHK are prismes, and that of equal height, as being placed between the parallel planes ABD, HIK, but
f 2. ax. 1. the base BFGE is f double of the base FDG. wherefore
g 40. 11. the said prismes are equal; whereof the one BFGEIH is greater than the pyramide BEFI, that is, than AEGH, the whole than its part; and consequently the two prismes are greater than the two pyramides and so exceed the half of the whole pyramide ABDC. *Which was to be dem.*

PROP. IV.

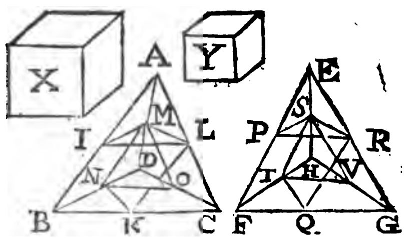


If there are two pyramides ABCD, EFGH, *of the same altitude, having triangular bases* ABC, EFG; *and either of them be divided into two pyramides* (AILM, MNOD; *and* EPRS, STVH) *equal one to the other and like to the whole; and into two equal prismes* (IBKLMN, KLCNMO; *and* PFQRST, QRGTSV;) *and if in like manner either of those pyrs. made by the former division be divided, and this be done continually; then as the base of one pyramide is to the base of the other pyramide, so are all the prismes which are in one pyramide, to all the prismes which are in the other pyramide, being equal in multitude.*

For (applying the construction of the precedent
a 15. 5. prop.) BC. KC a :: FG. QG. b therefore the triangle
b 22. 6. ABC is to the like triangle LKC as EFG is to c the
c 2. 6. &c. like RQG. therefore by permutation ABC. EFG d ::
d 16. 5. LKC. RQG e :: the prisme KLCNMO. QRGTSV
e sch. 34. 11 (for these are of equal altitude) f :: IBKLMN. PFQ-
f 1. 5. RST, g wherefore the triang. ABC. EFG :: the prisme
g 12. 5. KLCMNO + IBKLMN. the prisme QRGTSV + PFQRST. *Which was to be dem.* But

But if the pyramides MNOD, AILM; and EPRS, STVH be further divided; in like manner the four new prismes made hereby shall be to four produced before, as the bases MNO and AIL are to the bases STV, and EPR, that is, as LKC to RQG, or as ABC to EFG. *b* wherefore all the prismes of the pyramide ABCD are to all the prismes of the pyramide EFGH as the base ABC is to the base EFG. *Which was to be dem.*

PROP. V.

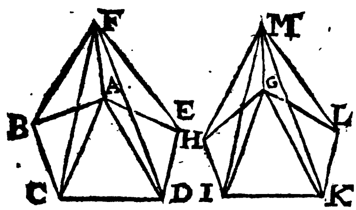


Pyramides ABCD, EFGH, *being of the same altitude, and having triangular bases* ABC, EFG, *are one to another as their bases* ABC, EFG.

Let the triangle ABC. EFG :: ABCD. X. I say X is equal to the pyramide EFGH. For if it be possible, let X be $\sqsupset$ EFGH. and let the excess be Y. Divide the pyramide EFGH into prismes and pyramides, and the other pyramides in like manner, *a* till the pyrs. left EPRS, STVH, be less than the solid Y. Therefore since the pyramide EFGH $=$ X + Y, it is manifest that the remaining prismes PFQRST, QRGTSV are greater than the solid X, Conceive the pyramide ABCD divided after the same manner; *b* then will be the prisme IBKLMN + KLCNMO. PFQRST + QRGTSV :: ABC. EFG *c* :: the pyr. ABCD. X. *d* therefore X $\sqsubset$ the prisme PFQRST + QRGTSV; which is contrary to that which was affirmed before. a 11 10. b 4. 12. c *hyp.* d 14. 5.

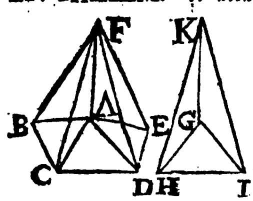
Again, conceive X $\sqsubset$ the pyr. EFGH. and make the pyr. EFGH, Y :: X. the pyr. ABCD *e* :: EFG. ABC. Because EFGH *f* $\sqsupset$ X, *g* thence Y $\sqsupset$ the pyr. ABCD. which is shewn before to be impossible. Therefore I conclude, that X is equal to the pyr. EFGH. *Which was to be demonstrated.* e *hyp. and cor.* 4. 5. f *suppos.* g 14. 5.

PROP. VI.



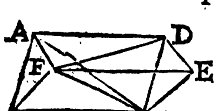
Pyramides ABCDEF, GHIKLM, *being of the same altitude, and having polygonous bases* ABCDE, GHIKL, *are to one another as their bases* ABCDE, GHIKL *are.*

Draw the right lines AC, AD, GI, GK. then is the base ABC. ACD *a* :: the pyr. ABCF. ACDF, *b* therefore by composition, ABCD. ACD :: the pyr. ABCDF. ACDF. *a* but also ACD. ADE :: the pyr. ACDF. ADEF. *c* therefore by equality ABCD ADE :: ABCDF. ADEF, and *b* thence by composition. ABCDE ADE :: the
a 5. 12. pyr. ABCDEF. ADEF, moreover ADE. GKL *d* :: the
b 18. 5. pyr. ADEF GKLM; and as before, and inversely GKL.
c 22. 5. GHIKL :: the pyr. GKLM. GHIKLM. *c* therefore a-
d 5. 12. gain by equality ABCDE. GHIKL :: the pyr. ABCDEF. GHIKLM. *Which was to be dem.*



If the bases have not sides of equal multitude, the demonstration will proceed thus. The base ABC. GHI *e*
e 5. 12. :: the pyr. ABCF. GHIK. *e* and ACD. GHI :: the pyr. ACDF. GHIK, *f* therefore the
f 24. 5. base ABCD. GHI :: the pyr. ABCDF. GHIK. *e* Moreover the base ADE. GHI :: the pyr. ADEF. GHIK. *f* therefore the base ABCDE. GHI :: the pyr. ABCDEF. GHIK.

PROP. VII.

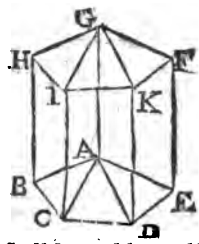


Every prisme, ABCDEF, *having a triangular base, may be divided into three pyrs.* ACBF, ACDF, CDFE, *equal one to the other, and having triangular bases.*

a 34. 1. Draw the diameters of the parallelograms AC, CF,
b 5 12. FD. Then the triangle ACB is *a* = ACD. *b* therefore the

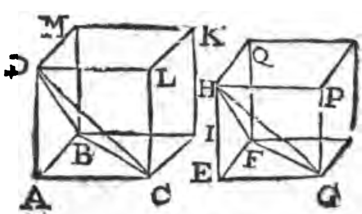
the pyramides of equal height ACBF, ACDF are equal. In like manner the pyr. DFAC = the pyr. DFEC, but ACDF and DFAC are one and the same pyr. *c* therefore the three pyramides ACBF, ACDF, DFEC, into which the prisme is divided, are equal one to the other. *Which was to be demonstrated.* c 1. ax. 1

Hence, every pyramide is the third part of the prisme that has the same base and height with it, or every prisme is treble of the pyramide that has the same base and height with it.



For resolve the polygonous prisme ABCDEGHIKF into triangular prismes; and the pyr. ABCDEH into triangular pyramides; *a* then all the parts of the prisme shall be treble to all the parts of the pyramide, *b* consequently the whole prisme ABCDEGHIKF is treble to the whole pyr. ABCDEH. *Which was to be dem.* a 7. 12. b 1. 5.

PROP. VIII.



Like pyramides ABCD, EFGH, *which have triangular bases* ABC, EFG *are in triplicate ratio of their homologous sides* AC, EG.

a Compleat the parallelpps. ABICDMKL, EFNGHQOP, which *b* are like, and *c* sextuple of the pyramides ABCD, EFGH. *d* and therefore the pyrs. have the same proportion to one another as the parallelpps. have, that is, *e* triplicate of their homologous sides. a 27. 11. b 9. def. 11. c 28. 11. and 7. 12. d 15. 5. e 33. 11.

Coroll.

Hence, also like polygonous pyramides are in triplicate ratio of their homologous sides; as may be easily prov'd by resolving them into triangular pyramides.

PROP. IX.

See the prec. Scheme.

In equal pyramides ABCD, EFGH, *having triangular bases* ABC, EFG, *the bases and altitudes are reciprocal; And pyramides having triangular bases, whose altitudes and bases are reciprocal, are equal.*

1. *Hyp* The compleated parallelpps. ABICDMKL,
a 28. 11. EFNGHQOP are *a* sextuple of the equal pyramides AB-
and 7. 12. CD, EFGH (each to each) and so equal one to the
b 34. 11. other, therefore the altitude (H.) the altitude (D) *b* ::
c 15. 5. ABIC. EFNG *c* :: ABC. EFG. *Which was to be dem.*
d *hyp.* 2 *Hyp.* The altitude (H.) the altitude (D) *d* :: ABC.
e 15. 5. EFG *e* :: ABIC EFNG *f* therefore the parallelpps.
f 34. 11. ABICDMKL, EFNGHQOP are equal, *g* consequent-
g 6. *ax.* 1. ly also the pyramides ABCD, EFGH being subsextuple of the same, are equal *Which was to be dem.*

The same is applicable to polygonous pyramides, for they may also in like manner be reduced to triangulars.

Coroll.

Whatsoever is dem. of pyramides in prop. 6, 8, 9 *does likewise agree to any sort of prismes; seeing they are triple of the pyramides that have the same base and altitude with them.* Therefore

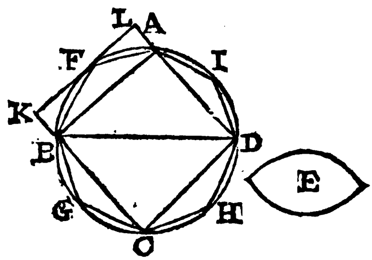
1. The proportion of prismes of equal altitude is the same with that of their bases.
2. The proportion of like prismes is triplicate of that of their homologous sides.
3. Equal prismes have their bases and altitudes reciprocal; and prismes which are so reciprocal; are equal.

Schol.

From what has been hitherto dem. the dimension of any prismes and pyramides may be collected.
a 1. *cor.* 12. *a* The solidity of a prisme is produced from the al-
& *sch.* 4. tude multiplied into the base; *b* and therefore likewise
11. that of a pyr. from the third part of the altitude mul-
b 7. 12. tiplied into the base.

PROP.

PROP. X.



Every Cone is the third part of a cylinder having the same base with it ABCD, *and the altitude equal.*

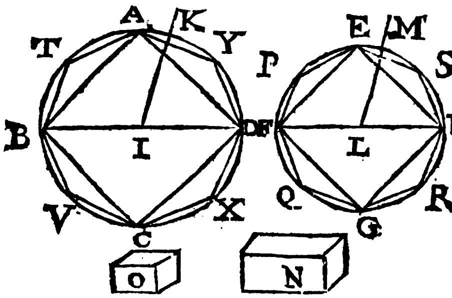
If you deny it, then first let such cylinder be more than triple to the cone, and let the excess be E. A prisme described on a square in the circle ABCD *a* is subduple of a prisme described upon a square about the circle, being equal to it and the cylinder in height. Therefore a prisme upon the square, ABCD exceeds the half of the cylinder, and likewise a prisme upon the base AFB, of equal height to the cylindars, *b* is greater than the half of the segment of the cyl. AFB, continue an equal bisection of the arches, and substract the prismes till the remaining segments of the cyl. namely, at AF, FB, &c. become less than the solid E. Therefore the cyl. — segm. AF, FB, &c. (the prisme on the base AFBGCHDI) *c* is greater than the cyl. — E (*d* the triple of the cone.) therefore the pyr. *e* a third part of the said prisme (being placed on the same base, and of the same height) is greater than the cone of equal height on the base ABCD a circle, *i. e.* the part greater than the whole. *Which is absurd.*

See the second figure of this Book.
a *sch.* 7. 4. *and cor.* 9. 12.
b *sch.* 27. 3. *and cor.* 9. 12.
c 5. *ax.* 1.
d *hyp.*
e *cor.* 7. 12.

But if the cone be affirmed to be greater than the third part of the cyl then let the excess be E. Detract the pyrs. from the cone, as you did in the first part the prismes from the cyl. till some segments of the cone remain, suppose at AF, FB, BG, &c. less than the solid E therefore the cone — E (*f* ⅓ of the cyl.) ⊐ the pyr. AFBGCHDI (the cone — seg. AF, FB, &c.) therefore the prisme triple to the pyr. (*viz.* of equal height, and on the same base) is greater than the cyl. on the base ABCD, the part than the whole. *Which is abs.* Wherefore it must be granted, that the cyl. is equal to triple of the cone. *Which was to be dem.*

PROP.

PROP. XI.



Cylinders and Cones ABCDK, EFGHM, *being of the same altitude, are to one another as their bases* ABCD, EFGH *are.*

Let the circle ABCD. the cir. EFGH :: the cone ABCDK. N. I say N is equal to the cone EFGHM.

For if it be possible, let N be ⊐ the cone EFGHM, and let the excess be O. The preparation and argumentation of the prec. prop. being supposed; then shall O be greater than the segments of the cone EP, PF, FQ, &c. and so the solid N ⊐ the pyr. EPFQGRHSM. In
a 30. 3. *and* 1. *post.* the circle ABCD *a* make a like polyg. fig. ATBVCXDY. Because the pyr. ABVYK. the pyr. EFQSM *b* :: the
b 6. 12. polygon ATBVY. the polygon EPFQS *c* :: the circle
c *cor.* 2. 12. ABCD. the cir. EFGH *d* :: the cone ABCDK. N. *e* thence
d *hyp.* the pyr. EPFQGRHSM shall be ⊐ N. contrary to
e 14. 5. what was affirmed before. Again conceive N ⊏ the cone EFGHM. and make the cone EFGHM. O :: N.
f *hyp & by inversion.* the cone ABCDK *f* :: the cir. EFGH. ABCD. *g* therefore O ⊐ the cone ABCDK; *which is absurd*, as ap-
g 14. 5. pears by what is shewn in the first part.

Therefore rather admit ABCD. EFGH :: the cone ABCDK. EFGHM. *Which was to be dem.*

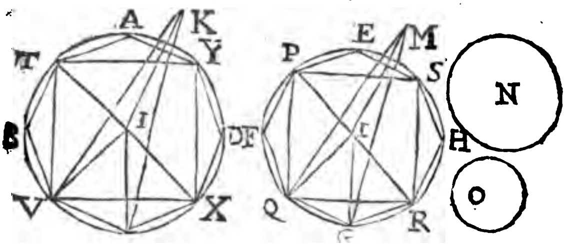
The same may be dem. of cylinders, if cylinders and prismes be conceived in the place of cones and pyramides. therefore, &c.

Schol.

Hence, is gathered the dimension of all sorts of cylinders and cones. The solidity of a right cyl. is produced
a 1. *Prop. de demens. cir.* from the circular base (*a* the dimension whereof is to be learnt out of *Archimedes*) multiplied into the height;
b 11. 12. *b* whence in like manner that of every cylinder.
c 10. 12. *c* Therefore the solidity of a cone is produced from the third part of the altitude multiplied into the base.

Like

PROP. XII.



Like cones and cylinders ABCDK, EFGHM, *are in triplicate ratio of that of the diameters* TX, PR, *of their bases* ABCD, EFGH.

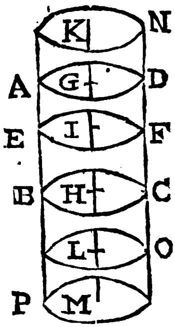
Let the cone A have to N a triplicate ratio of TX to PR. I say N is $=$ the cone EFGHM For if it be possible let N be $\sqsupset$ EFGHM, and let the excess be O, therefore N $\sqsupset$ the pyr EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right lines VK, CK, VI, CI, and QM, GM, QL, GL. Because the cones are like, *a* thence VI. IK :: QL, LM. but the angles VIK, a 24 *def*. 11
QLM *b* are right angles, *c* therefore the triangles VIK, b 18 *def*. 11
QLM are equiangular, *d* whence VC. VI :: QG QL. c 6. 6.
also VI. VK :: QL QM. therefore by equality VC VK d 4. 6.
:: QG QM. *e* moreover VK. CK :: QM MG. there- e 7. 5.
fore again by equality VC. CK :: QG. GM, *f* therefore f 5. 6.
the triangles VKC, QMG are like; and by a like way of reasoning the other triangles of this pyr. are like to the other of that, *g* wherefore the pyrs. themselves g 9. *def*. 11.
are like *h* But these are in triplicate proportion of that h *cor* 8. 12.
of VC to QG, *k* that is, of VI to QL, *l* or TX to PR. k 4 6.
m therefore the pyr. ATBVCXDYK. the pyr. EPFQ- l 15. 5.
GRHSM :: the cone ABCDK N. *n* whence the pyr. m *hyp and*
EPFQGRHSM $\sqsupset$ N. which is repugnant to what was 11 5.
affirmed before. n 14 5.

Again, take N $\sqsubset$ the cone EFGHM make the cone EFGHM. O :: N. the cone ABCDK *o* :: the pyr. EPRM. o *before &*
ATCK *p* :: GQ. VC thrice :: *q* PR. TX thrice, there- *inversly*.
fore O *r* is $\sqsupset$ ABCDK. which was before shewn to be p *cor*. 8. 12.
repugnant. Wherefore N $=$ the cone EFGHM. *Which* q 4. 6.
was to be dem. r 14 5.

But forasmuch as what proportion soever cones have, also cylinders, being triple of them, have the same; therefore cyl. shall be to cyl. in triplicate ratio of the diameters of their bases.

PROP.

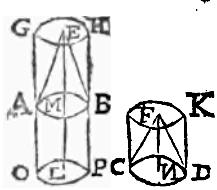
PROP. XIII.



If a cyl. ABCD *be divided by a plane* EF *parallel to the opposite planes* BC, AD, *then as one cyl.* AEFD *is to the other cyl.* EBCF, *so is the axis* GI *to the axis* IH.

a 3. 1. The axis being produced, *a* take GK = GI, and HL = IH = LM. and conceive planes drawn at the points K, L, M, parallel to the cir-
b 11. 12. cles AD, BC, *b* therefore the cyl. ED = the cyl. AN, and the cyl. EC *b* = BO *b* = OP. therefore the cyl. EN is the same multiple of the cyl. ED as the axis IK is of the axis IG, and in like manner the cyl. EP is the same multiple of the cyl. BF, as the axis IM is of the axis IH. but as IK is =, ⊏, ⊐ IM,
c 11. 12. *c* so is the cyl. EN =, ⊏, ⊐ EP. *d* therefore the cyl.
d 6. *def.* 5. AEFD. the cyl. EBCF :: GI. IH. *Which was to be dem.*

PROP. XIV.

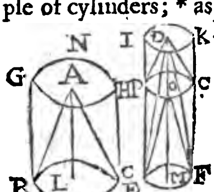


Cones AEB, CFD, *and cylinders* AH, CK, *insisting upon equal bases* AB, CD, *are to one another as their altitudes* ME, NF.

The cyl. HA, and the axis EM being produced, take ML = FN; and thro' the point L draw
a 11. 12. a plane parallel to the base AB, *a* then shall the cyl. AP
b 13. 12. be = CK. *b* but the cyl. AH, AP. (CK) :: ME. ML (NF.) *Which was to be dem.*

* *apply* 9, The same may be affirmed of cones which are subtri-
and 7. 12. ple of cylinders; * as also of prismes and pyramides.

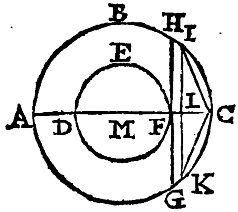
PROP. XV.



In equal cones BAC, EDF *and cylinders* BH, EK, *the bases and altitudes are recipro.* (BC. EF :: MD. LA.) *And cones and cylinders, whose bases and altitudes are reciprocal, are equal one to the other.*

a 14. 12. If the altitudes be equal then the bases are equal too,
b *constr.* and the thing is evident. If unequal, then take away
c *hyp.* MO = LA.
d 11 12. 1. *Hyp.* Then is MD. MO (*a* LA) *b* :: the cyl. EK. (*c*
e *hyp.* BH) EQ *d* :: the cir. BC EF. *Which was to be dem.*
f 14 12. 2. *Hyp.* BC. EF *e* :: DM. OM (LA) *f* :: the cyl, EK,
g 11. 5. EQ *g* :: BC. EF *h* :: BH. EQ. *k* Therefore the cyl. EK
h 11. 12. = BH. *Which was to be dem.*
k 9 5. The same argument may be used for cones. Two

PROP. XVI.



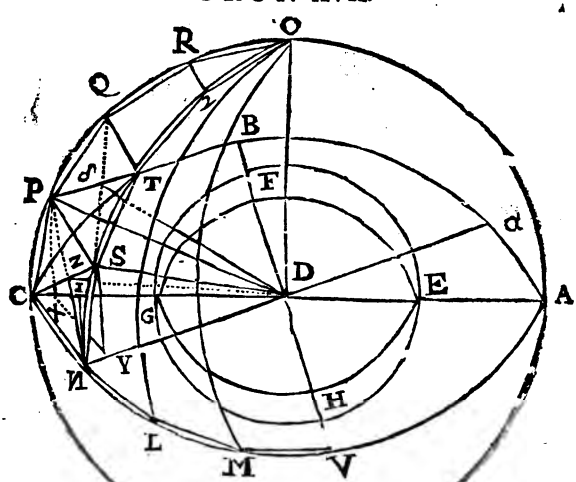
Two unequal circles ABCG, DEF, *having the same center* M, *to inscribe in the greater cir.* ABCG *a polygonous fig. of equal and even sides, which shall not touch the lesser cirle* DEF.

Thro' the center M draw the line AC cutting the cir. DEF in F, from whence raise a perpendicular FH. *a* divide the semicircle ABC into two equal parts; and the half thereof BC also; and so do continually, *b* till the arch IC becomes less than the arch HC. from I let fall the perpendicular IL. It is manifest that the arch IC measures the whole circle, and that the number of arches is even, and so that the subtended line IC is the side *c* of the polygon that may be inscribed without touching the lesser circle DEF. For HG *d* touches the circle DEF, *e* to which IK is parallel, and placed outwardly; *f* wherefore IK does not touch the circle DEF ; much less do CI, CK, and the other sides of the polygon more remote from the center. *Which was to be done.*

a 30. 3.
b 1. 10.
c *sch.* 16. 4.
d *cor.* 16. 3.
e 28. 1.
f 34. *def* 1.

Coroll. Observe that IK touches not the circle DEF.

PROP. XVII.



Two

Two ſpheres ABCV, EFGH, *conſiſting about the ſame center* D, *being given, to inſcribe a ſolid of many ſides* (*or Polyedron*) *in the greater ſphere* ABCV, *which ſhall not touch the ſuperficies of the leſſer ſphere* EFGH.

Let both the ſpheres be cut by a plane paſſing thro'
the center, making the circles EFGH, ABCV; and the
diameters AC, BV drawn, cutting derpendicularly. In
a 16. 12. the circle ABCV, *a* inſcribe the equilateral polygon
VMLNC, &c. not touching the circle EFGH: Then
draw the diameter Nα, and erect DO perpendicular
to the plane ABC thro' DO, and thro' the diameters
AC, Nα, conceive planes DOC, DON erected, which
b 18. 11. ſhall be *b* perpendicular to the circle ABCV, and ſo in
c cor. 33. 6. the ſuperficies of the ſphere make *c* the quadrants DOC,
DON. In which let the right lines CP, PQ, QR, RO,
d 4. 1. NS, ST, Tγ, γO *d* be fitted, equal, and of equal multi-
tude with CN, NL, &c. make the ſame conſtruction in
the other quadrants OL, OM, &c. and in the whole
ſphere. Then I ſay the thing required is done.

From the points P, S, to the plane ABCV draw the
e 38. 11. perpendiculars PX, SY, *e* which ſhall fall on the ſec-
f 12. ax. tions AC, Nα Therefore becauſe both *f* the right an-
g 27. 3. gles PXC, SYN, *g* and PCX, SNY inſiſting on *h* equal
h 32. 1. circumferences, *f* are equal, the triangles alſo PCX,
k conſtr. SNY *h* are equiangular. Wherefore ſince PC *k* = SN,
l 26. 1. *l* alſo is PX = SY, *l* and XC = YN; *m* whence DX
m 3. ax. 1. = DY, *n* and therefore DX. XC :: DY. YN. *o* there-
n 7. 5. fore YX, NC are parallels, but becauſe PX, SY are e-
o 2. 6. qual, and ſince being perpendicular to the ſame plane
p 6. 11. ABCV, they are alſo *p* parallels, *q* therefore YX, SP
q 33. 1. ſhall be equal and parallels, *r* whence SP, NC, are pa-
r 9. 11. rallel one to the other; and ſo the *ſ* quadrilateral NC-
ſ 7. 11. PS, and for the ſame reaſon SPQT, TQRG, as alſo the
t 2. 11. *t* triangle γRO are ſo many planes In like manner the
whole ſphere may be ſhewn full of ſuch quadrilaterals
and triangles, wherefore the figure inſcribed is a po-
lyedron.

u 11. 11. From the center D *u* draw DZ perpendicular to the
plane NCPS; and join ZN, ZC, ZS, ZP. Becauſe DN.
x 4. 6. NC *x* :: DY. YX, thence NC is *y* $\sqsubset$ YX (SP.) and in
y 14. 5. like manner SP $\sqsubset$ TQ, and TQ $\sqsubset$ γR. And becauſe
z 3. def. 11. the angles DZC, DZN, DZS, DZP *z* are right, and
a 15. def. 1. the ſides DC, DN, DS, DP, *a* equal, and DZ common,
b 47. 1. *b* thence ZC, ZN, ZS, ZP are equal one to the other;
c 15. def. 1. and conſequently about the quadrilateral NCPS, *c* a cir-
cle

cle may be described, in which (because NS, NC, CP, are *d* equal, and NC $\sqsubset$ SP) NC *e* subtends more than a quadrant, *f* therefore the angle NZC at the center is obtuse, *g* therefore NCq $\sqsubset$ 2 ZCq (ZCq + ZNq) Let NI be drawn perpendicular to AC, therefore since the angle ADN (*h* DNC + DCN) *k* is obtuse, the half of it DCN shall be greater than the half of a right angle; and so that which remains of the right angle CNI shall be less than it, *n* whence IN $\sqsubset$ IC, therefore NCq (NIq + ICq) *o* $\sqsupset$ 2 INq. therefore IN $\sqsubset$ ZC, and consequently DZ *p* $\sqsubset$ DI. but the point I is *q* without the sphere EFGH. and so, much more, the point Z. wherefore the plane NCPS, (of which *r* the nearest point to the center is Z,) does not touch the sphere EFGH. And if a perpendicular Dδ be drawn to the plane SPQT, the point δ, and so also the plane SPQT is yet further removed from the center, which is also true of the other planes of the polyedron. Therefore the polyedon ORQPCN, &c. inscribed in the greater sphere, does not touch the lesser. *Which was to be done.*

d *constr.*
e 28. 3.
f 33. 6.
g 12. 2.
h 32. 1.
k 9 *ax.* 1.
l 5. 1.
n 19. 1.
o 47. 1.
p 47. 1.
q *cor.* 16. 12.
r 47. 1.

Coroll.

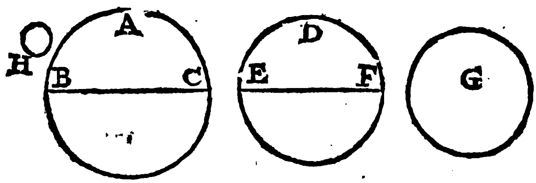
Hence it follows, *that if in any other sphere a solid polyedron, like to the abovesaid solid polyedron, be inscribed, the proportion of the polyedron in one sphere to the polyedron in the other is triplicate of that of the diameters of the Spheres.*

For if right lines be drawn from the centers of the spheres to all the angles of the bases of the said polyedrons, then the polyedrons will be divided into pyrs. equal in number and like; whose homo. sides are semidiameters of the spheres; as appears, if the lesser of these spheres be conceived described within the greater about the same center. For the right lines drawn from the center of the sphere to the angles of the bases will agree one to the other by reason of the likeness of the bases; and so will like pyramides be made. Wherefore since every pyr. in one sphere to every pyr. like it in the other sphere *a* has proportion triplicate to that of the homologous sides, that is, of the semidiameters of the spheres; and *b* as one pyr. is to one pyr so all the pyrs. that is, the solid polyedron composed of these, are to all the pyrs. that is, the solid polyedron composed of the others; therefore the polyedron of one sphere shall have to the polyedron of the other sphere, proportion triplcate of that of the semidiameters, *c* and so of the diameters of the spheres.

a *cor.* 8. 12.
b 12. 5.
c 15. 5.

PROP.

PROP. XVIII.



Spheres BAC, EDF, *are in triplicate ratio of their diameters* BC, EF.

Let the ſphere BAC be to the ſphere G in tripli. proportion of that of the diameter BC to the diameter EF. I ſay G = EDF. For if it be poſſible, let G be $\sqsupset$ EDF. and conceive the ſphere G concentrical with EDF. In the ſphere EDF *a* inſcribe a polyedron not touching
a 17. 12. the ſphere G, and a like polyedron in the ſphere BAC.
b *cor* 17.12. Theſe polyedrons *b* are in triplicate proportion of the
c *hyp.* diameters BC, EF, *c* that is, of the ſphere BAC to G.
d 14. 5. *d* conſequently the ſphere G is greater than the polyedron inſcribed in the ſphere EDF, the part than the whole.

Again, if it be poſſible, let the ſphere G be $\sqsubset$ EDF. and as the ſphere EDF is to another ſphere H, ſo let G
e *hyp. inverſe.* be to BAC, *e* that is, in triplicate proportion of the diameter EF to BC, therefore ſince BAC *f* $\sqsubset$ H, we ſhall
f 14. 5. incur the abſurdity of the firſt part, wherefore rather the ſphere G = EDF. *Which was to be dem.*

Coroll.

Hence, as one ſphere is to another ſphere, ſo is a polyedron deſcribed in that to a like polyedron deſcribed in this.

The end of the twelfth Book.

THE

THE THIRTEENTH BOOK OF *EUCLIDE's* ELEMENTS.

PROPOSITION I.

If a right line z *be divided according to extreme and mean proportion* (z. a : : a. e.) *the square of the half of the whole line* z, *and of the greater segment* a, *as one line, is quintuple to that which is described of half of that whole line* z.

I say Q. $a + \frac{1}{2} z = 5$ Q: $\frac{1}{2}$ z. *a* that is, $aa + \frac{1}{4} zz +$ $za = zz + \frac{1}{4} zz$ *b* or $aa + za$ $= zz$. For $ze + za$ *c* $= zz$. and ze *d* $= aa$. *e* therefore $aa +$ $za = zz$. *Which was to be demonstrated.*

a 4. 2.
b 3. *ax.* 1.
c 2. 2.
d *hyp. and* 16. 6.
e 2. *ax. and* 1. *ax.*

PROP. II.

See the 1st Scheme.

If a right line $\frac{1}{2}$ $z + a$ *be in power quintuple to a segment of it self* $\frac{1}{2}$ z. *the line double of the said segment* (z) *being divided according to extreme and mean proportion, the greater segment is* (a) *the other part of the right line at first given* $\frac{1}{2}$ $z + a$.

* 4. 2. I say z. a :: a. e. Because by the hyp. * $aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$; or $aa + za = zz$ a $= ze + za$, b thence
a 2. 2. shall aa be $= ze$. c wherefore z. a :: a. e. *Which was to be demonstrated.*
b 3. ax. 1.
c 17. 6.

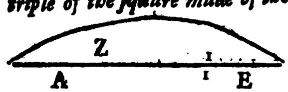
PROP. III.

If a right line z *be divided according to extreme and mean proportion* (z. a :: a. e.) *the line made of the less segment* e *and half of the greater segment* a, *is in power quintuple to the square, which is described of the half line of the greatest segment* a.

a 4. 2. I say $Q: e + \frac{1}{2}a = 5\,Q: \frac{1}{2}a$: *a* that is $ee + \frac{1}{4}aa + ea = aa + \frac{1}{4}aa$. *b* or $ee + ea = aa$. For $ee + ea$ *c* $= ze$ *d* $= aa$. *Which was to be demonstrated.*
b 3. ax.
c 3. 2.
d hyp. and 17. 6.

PROP. IV.

If a right line z *be cut according to extreme and mean proportion* (z a :: a. e) *the square made of the whole line* z, *and that made of the lesser segment* e, *both together, are triple of the square made of the greater segment* a.



a 4. 1. I say $zz + ee = 3\,aa$. *a* or $aa + ee + 2\,ae + ee = 3\,aa$. For $ae + ee$ *b* $= ze$ *c* $= aa$. *d* therefore $aa + 2\,ae + 2\,ee = 3\,aa$. *Which was to be demonstrated.*
b 3. 2.
c 17. 6.
d 2. ax.

PROP. V.

D———A———C———B *If a right line AB be cut according to extreme and mean proportion in C, and a line AD, equal to the greater segment BC, added to it, the whole right line DB is divided according to extreme and mean proportion; and the greater segment is the right line AB given at the beginning.*

a hyp. For because AB. AD *a* :: AC. CB. and by inversion AD. AB :: CB. AC. therefore by composition DB. AB :: AB. AC (AD.) *Which was to be demonstrated.*

Schol.

But if BD.BA :: BA. AD. then shall be BA.AD. :: AD. BA—AD. For by dividing BD—BA (AD) BA :: BA—AD. AD. therefore inversely BA. AD :: AD. BA — AD.

PROP.

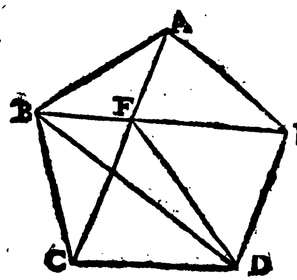
PROP. VI.

D———A———C———B *If a rational right line AB be cut according to extreme and mean proportion in C, either of the segments (AC, CB) is an irrational line of that kind which is called apotome or residual.*

To the greater segment AC *a* add AD = ½ AB. *b* therefore DCq = 5 DAq. *c* therefore DCq ⊓⊔ DAq. consequently *d* since AB, *e* and so the half thereof DA are ῥ, likewise DC is ῥ. But because 5. 1 :: not Q.Q. *f* thence is DC ⊓⊔ DA. *g* therefore DC — AD, that is, AC, is a residual line. Further, because ACq *h* = AB × BC, and AB is ῥ, *i* likewise BC is a residual line. *Which was to be demonstrated.*

a 3. 1.
b 1. 13.
c 6. 10.
d *hyp.*
e *sch.* 12. 10.
f 9. 10.
g 74. 10.
h 17. 6.
i 98. 10.

PROP. VII.



If three angles of an equilateral pentagone ABCDE, whether they follow in order, (EAB, ABC, BCD,) or not, (EAB, BCD, CDE) are equal, the pentagone ABCDE shall be equiangular.

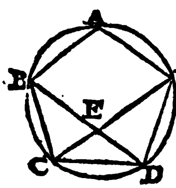
Let the right lines BE, AC, BD, be subtended to the equal angles in order.

Because the sides EA, AB, BC, CD, and the included angles *a* are equal, *b* therefore shall the bases BE, AC, BD, *c* and the angles AEB, ABE, BAC, BCA, be equal. *d* Wherefore BF = FA, *e* and consequently FC = FE; therefore the triangles FCD, FED, are equilateral one to the other: *f* whence the angle FCD = FED. *g* consequently the angle AED = BCD. In like manner the angle CDE is equal to the rest; wherefore the pentagone is equiangular. *Which was to be demonstrated.*

a *hyp.*
b 4. 1.
c 4. and 5. 1.
d 6. 1.
e 3. *ax.* 1.
f 8. 1.
g 2. *ax.* 1.

But if the angles EAB, BCD, CDE, which are not
h 4. 1. in order, be supposed equal, *h* then shall the angle AEB
k 5. 1. be=BDC, and BE=BD. *k* and thence the angle BED
l 2. *ax.* =BDE. *l* consequently the whole angle AED=CDE, therefore because the angles A, E, D, in order, are equal, as before, the pentagone shall be equiangular. *Which was to be demonstrated.*

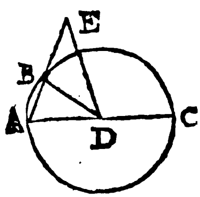
PROP. VIII.



If in an equilateral and equiangular pentagone ABCDE, two right lines BD, CE, subtend two angles BCD, CDE following in order, those lines do cut one another according to extreme and mean proportion; and their greater segments BF or EF are equal to the side of the pentagone BC.

a 14. 4. *a* Describe about the pentagone the circle ABD. *b*
b 28. 3. The arch ED is=BC, *c* therefore the angle FCD=FDC. *d*
c 27. 3. therefore the angle BFC=2 FCD (FCD+FDC.) But the
d 32. 1. arch BAE is=2 ED, and consequently the angle BCF. *e*
e 33. 6. =2 FCD=BFC. *f* wherefore BF=BC. *Which was to*
f 6. 1. *be demonstrated.* Moreover, because the triangles BCD,
g 27. 3. FCD, are *g* equiangular. *h* therefore BD. DC (BF.) ::
h 4. 6. CD. (BF.) FD. and likewise EC. EF :: EF. FC. *Which was to be demonstrated.*

PROP. IX.



If the side of an Hexagone BE, and the side of a Decagone AB both described in the same circle ABC, be added together, the whole right line AE is cut according to extreme and mean proportion (AE. BE :: BE. AB) and the greater segment therefore is the side of the Hexagone BE.

Draw the diameter ADC, and join the right lines
a *hyp.* and DB, DE. Because the angle BDC *a*=4 BDA and the
27. 3. angle BDC *b*=2 DBA (DAB+DBA) thence shall
b 32. 1. DBA (*b* DBE+BED) *c* be=2 BDA *d*=2 BDE, whence
c 7. *ax.* 1. the angle DBA or DAB *e*=ADE. Therefore the tri-
d 5. 1. angles ADE, ADB, are equiangular: *f* wherefore AE.
e 1. *ax.* 1. AD (*g* BE) :: AD. (BE.) AB. *Which was to be demon-*
f 4. 6. *strated.*
g *cor.* 15. 4.

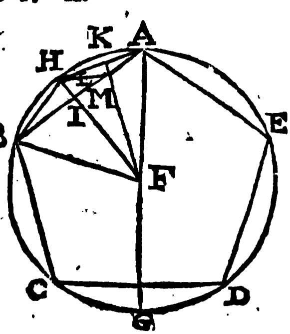
Coroll.

Coroll.

Hence, If the side of a hexagone in a circle be cut according to extreme and mean proportion; the greater *sch.* 5. 13.
segment thereof shall be the side of the decagone in the same circle.

PROP. X.

If an equilateral Pentagone ABCDE be inscribed in a circle ABCE, the side of the pentagone AB containeth in power both the side of a hexagone FB, and the side of a decagone AH inscribed in the same circle.



Draw the diameter AG, and bisect the arch AH in K, and draw FK, FH, FB, BH, HM.

The semicircle AG — the arch AC *a* = AG — AD. a 28. 3. *and*
that is, the arch CG = GD *b* = AH = HB. therefore the 3. *ax.*
arch BCG = 2 BHK; *c* and so the angle BFG = 2 BFK. b *hyp. and*
d but the angle BFG = 2 BAG. *e* therefore the angle 7. *ax.*
BFK = BAG. Wherefore the triangles BFM, FAB, *f* c 33. 6.
are equiangular. *g* whence AB, BF :: BF. BM. *h* there- d 20. 3.
fore AB × BM = BFq. Moreover the angle AFK *k* = e 1. *ax.* 1.
HFK, and FA = FH. *m* wherefore AL = LH, *m* and the f 32. 1.
angles FLA, FLH are equal, and so right angles, there- g 4. 6.
fore the angle LHM *m* = LAM *n* = HBA. therefore the h 17. 6.
triangles AHB, AMH, *o* are equiangular; wherefore k 27. 3.
AB. AH :: AH. AM. *q* therefore AB × AM = AHq. m 4. 1.
Since therefore ABq *r* = AB × BM + AB × AM, *s* thence n 27. 3.
ABq = BFq + AHq. *Which was to be demonstrated.* o 32. 1.

Coroll. p 4. 6.

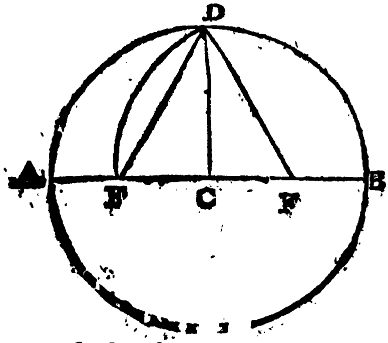
1. Hence, a right line (FK) which being drawn q 17. 6.
from the center (F) divides an arch (HA) into two equal r 2. 2.
segments, does also divide the right line (HA) sub- s 2. *ax.*
tending that arch perpendicularly into two equal segments.

2. The diameter of a circle (AG.) drawn from any angle (A) of a pentagone, does divide equally in two, both the arch (CD,) which the side of the pentagone opposite to that angle subtends, and also the opposite side it self (CD) and that perpendicularly.

Schol.

Here, according to our promise, we shall lay down a ready praxis of the 11th prop. *of the* 4th *Book.*

Probl.



To find out the side of a pentagone to be inscribed in a circle ADB.

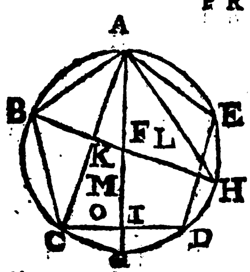
Draw the diameter AB, to which erect a perpendicular CD at the center C, divide CB equally in E, and make EF = ED. then DF shall be the side of the pentagone.

a 6. 2. For $BF \times FC + ECq$ *a* $= EFq$ *b* $= EDq$ *c* $= DCq +$
b *constr.* ECq. *d* therefore $BF \times FC = DCq$ or BCq. *e* wherefore
c 47. 1. BF. BC. :: BC. FC. therefore since BC is the side of a
d 3. *ax.* hexagone, *f* FC shall be the side of a decagone. Con-
e 17. 6. sequently DF *b* $= \sqrt{DCq + FCq}$ *g* is the side of a pen-
f 9. 13. tagone. *Which was to be done.*

g 10. 13.

h 47. 1.

PROP. XI.



If in a circle ABCD, whose diameter AG, is rational, an equilateral pentagone be inscribed ABCDE; the side of the pentagone AB is an irrational line of that kind which is called a minor line.

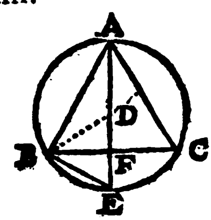
Draw the diameter BFH,
* 10. 6. and the right lines AC, AH; and * make FL = ¼ of the ra-
a *cor.* 10. dius FH; and CM = ¼ CA.
13. Because the angles AKF, AIC, are *a* right angles, and
b 32. 1. CAI common, the triangles AKF, AIC, are *b* equian-
c 4. 6. gular; *c* therefore CI. FK *c* :: CA. FA (FB) *d* :: CM.
d 15. 5. FL.

FL. therefore by permutation FK. FL :: CI. CM d :: CD. CK (2 CM) and so by e composition CD + CK. CK :: KL. FL. f consequently Q : CD + CK. (g 5 CKq) CKq :: KLq. FLq. therefore. KLq = 5 FLq. wherefore if BH (ρ) be taken 8, FH shall be 4, FL 1, and FLq 1, BL 5, and BLq 25, KLq 5. by which it appears that BL and KL are ρ h ⊓⊔, k and so BK is a residual, and KL its congruent or adjoining line. but since BLq − KLq. = 20. l thence BL ⊓⊔ √ BLq − KLq * whence BK shall be a fourth residual line. Therefore because ABq m is = HB × BK, n shall AB be a minor line. *Which was to be demonstrated.*

e 18. 5.
f 22. 6.
g 1. 13.
h 9. 10.
k 74. 10.
l 9. 10.
*4 def. 85. 10.
m cor. 8. 6. and 17. 6.
n 95. 10.

PROP. XII.

If in a circle ABEC an equilateral triangle ABC be inscribed, the side of that triangle AB is in power triple to the line AD drawn from D the center of the circle to the circumference.



The diameter being extended to E, draw BE. Because the arch BE a = EC, the arch BE is the sixth part of the circumference, b therefore BE = DE. hence AEq c = 4 DEq (4 BEq) d = ABq + BEq (+ ADq.) e consequently ABq = 3 ADq, *Which was to be demonstrated.*

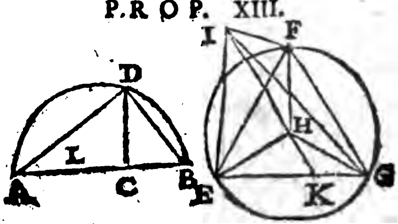
a cor. 10. 13.
b cor. 15. 4.
c 4. 2.
d 47. 1.
e 3. ax. 1.

Coroll.

1. AEq. ABq :: 4. 3.
2. ABq. AFq :: 4. 3. f For ABq. AFq :: AEq. ABq.
3. DF = FE. For the triangle EBD g is equilateral, h and BF perpendicular to ED. h therefore EF = FD.
4. Hence, AF = DE + DF = 3 DF.

f cor. 8. 6. and 22. 6.
g cor. 15. 4.
h cor. 3. 3.

PROP. XIII.



To describe a pyramid EGFI, and comprehend it in a sphere given: and to demonstrate that the diameter of the sphere AB is in power sesquialter of the side EF of the pyramid EGFI.

a 10. 6. About AB describe the semicircle ADB; *a* and let AC be = 2 CB. From the point C erect the perpendicular CD, and join AD, DB, then at the interval of the radius HE = CD describe the circle HEFG, *b* wherein in-
b *cor.* 15. 4.
c 12. 11. scribe the equilateral triangle EFG. from H *c* erect IH = CA perpendicular to the plane EFG, produce IH to
d 3. 1. K, *d* so that IK = AB; and join the right lines IE, IF, IG. Then EFGI shall be the pyramid required.

e *constr.* For because the angles ACD, IHE, IHF, IHG, *e* are right angles; and CD, HE, HF, HG *e* equal, *e* and
f 4. 1. IH = AC; *f* therefore AD, IE, IF, IG, shall be equal
g 20. 6. among themselves. But because AC (2 CB.) CB *g* :: ACq.
h 2. *ax.* CDq. thence shall ACq be = 2 CDq. therefore ADq
k 12. 13. *f* = ACq + CDq *h* = 3 CDq = 3 HEq *k* = EFq. *l* there-
l 1. *ax.* 1. fore AD, EF, IE, IF, IG are equal, and so the pyramid EFGI is equilateral. But if the point C be placed upon H, and
m 8. *ax.* 1. AC upon HI, the right lines AB, IK, *m* shall agree, as being equal. Wherefore the semicircle ADB being drawn
n 15. *def.* 1. about the axis AB or IK *n* shall pass by the points E, F, G,
* 31. *def.* * and so the pyramid EFGI shall be inscribed in a sphere.
11. *Which was to be done.*
o *cor.* 8. 6. Also it is manifest that BAq. ADq *o* :: BA. AC *p* :: 3. 2.
p *constr.* *Which was to be demonstrated.*

Coroll.

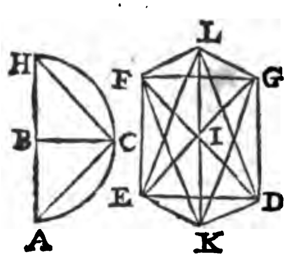
1. ABq. HEq :: 9. 2. For if ABq be put 9, then
q 12. 13. ADq (EFq) shall be 9. *q* consequently HEq shall be 2.

2. If L be the center, then shall AB. LC :: 6. 1. For
r *constr.* if AB be put 6, then AL shall be 3. *r* and thence AC 4. wherefore LC shall be 1. Hence,

3. AB HI :: 6. 4 :: 3. 2. whence

4. ABq. HIq :: 9. 4.

PROP. XIV.



To describe an Octaedron KEFGDL, and comprehend it in the given sphere, wherein a pyramid is: and to demonstrate that AH, the diameter of the sphere, is in power double of AC, the side of that Octaedron.

About AH describe the semicircle ACH. and from the center B erect the perpendicular BC. draw AC, HC;

then upon ED=AC *a* make the square EFGD, whose diameters DF, EG, cut in the center I. from I. draw IL=AB *b* perpendicular to the plane EFGD. produce IL *c* till IK=IL. and join KE, KF, KG, KD, LE, LF, LG, LD; then shall KEFGDL be the Octaedron required. a 46. 1. b 12. 11. c 3. 1.

For AB, BH, FI, IE, &c. being semidiameters of equal squares are equal one to the other. *d* whence the bases LF, LE, FE, &c. of the right angled triangles LIE, LIF, FIE, &c. are equal, and consequently the eight triangles LFE, LFG, LGD, LDE, KEF, KFG, KGD, KDE, are equilateral, *e* and make an Octaedron, which may be inscribed in a sphere, whose center is I. and IL or AB the radius. (because AB, IL, IF, IK, &c. *f* are equal.) *Which was to be done.* Moreover, it is evident that AHq (LKq) *g* = 2 ACq (2 LDq.) *Which was to be demonstrated.* d 4. 1. e 27. def. 11. f constr. g 47. 1.

Coroll.

1. Hence it is manifest, that in the Octaedron the three diameters EG, FD, LK do cut one the other perpendicularly in the center of the sphere.

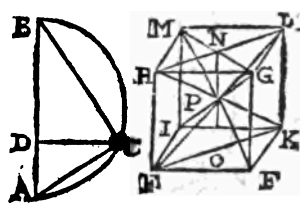
2. Also, that the three planes EFGD, LEKG, LFKD are squares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal pyramids EFGDL, and EFGDK, whose common base is the square EFGD.

4. Lastly, it follows that the opposite bases of the Octaedron are parallel one to the other. 15. 11.

PROP. XV.

To describe a cube EFGHIKLM, and comprehend it in the same sphere, wherein the former figures were; and to demonstrate that AB the diameter of the sphere is in power triple to EF the side of that cube.



Upon AB describe a semicircle ACB; *a* and make AB=3 DA, from D raise the perpendicular DC, and join BC and AC. Then upon EF=AC *b* make the square EFGH, upon whose plane let the right lines EI, FK, HM, GL, stand perpendicular, being equal to EF, and a 10. 6. b 46. 1.

and connect them with the right lines IK, KL, LM, IM, The solid EFGHIKLM, is a cube, as is sufficiently apparent from the construction.

In the opposite squares EFKI, HGLM, draw the diameters EK, FI, HL, MG, through which let the planes EKLH, FIMG be drawn, cutting one another in the line NO. which *c* shall divide equally in two parts the diameters of the cube EL, FM, GI, HK, in P the center of the cube. *d* therefore P shall be the center of a sphere passing through the angular points of the cube. Moreover ELq *e*= EKq + KLq *e*= 3 KLq, *f* or 3 ACq. but ABq. ACq *g*:: BA. DA *f*:: 3. 1. *h* therefore AB = EL. wherefore we have made a cube, &c. *Which was to be done.*

c cor. 39. 11.
d 15. def. 1, and 14. def. 11.
e 47. 1.
f constr.
g cor. 8. 6.
h 14. 5.

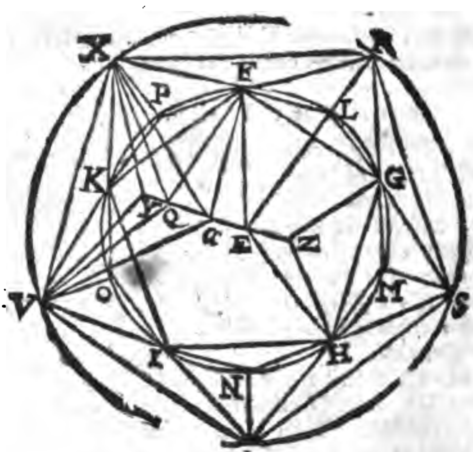
Coroll.

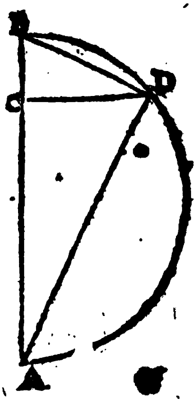
1. Hence it is manifest that all the diameters of the cube are equal one to another, and do equally bisect one another in the center of the sphere. And for the same reason the right lines which conjoin the centers of the opposite squares are bisected in the same center.

2. The diameter of a sphere containeth in power [the side of a tetraedron and of a cube, *viz.* ABq *k*= *l* BCq + *m* ACq.

k 47. 1.
l 13. 13.
m 15. 13.

PROP. XVI.





To describe an Icosaedron ZGHIK FTVXRST, and encompass it in the sphere, wherein were contained the foresaid solids; and to demonstrate that FG the side of the Icosaedron is that irrational line, which is called a minor line.

Upon AB the diameter of a sphere describe the semicircle ADB; and *a* make AB = 5 BC. then from C erect CD perpendicular, and draw AD and BD. At the distance EF = BD describe the circle EFKNG; *b* wherein inscribe the equilateral pentagone FKIHG. Divide equally in two parts the arches FG, GH, &c. and join the right lines FL, LG, &c. being the sides of a decagone. Then *c* erect EQ, LR, MS, NT, OV, PX equal to EF, and perpendicular to the plane FKNG; and connect RS, ST, TV, VX, XR; as also FX, FR, GR, GS, HS, HT, IT, IV, KV, KX. Lastly, produce EQ, and take QY = FL, and EZ = FL, and conceive the right lines ZG, ZH, ZI, ZK, ZF to be drawn; as also YV, YX, YR, YS, YT. Then I say the Icosaedron required is made.

a 10. 6.

b 11. 4.

c 12. 11.

For because EQ, LR, MS, NT, OV, PX, are *d* equal and *e* parallel, also those lines that join them EL, QR, EM, QS, EN, QT, EO, QV, EP, QX, *f* are equal and parallel. And thence likewise LM (or FG) RS, MN, ST, &c. are equal one to the other. *g* therefore the plane drawn through EL, EM, &c. is equidistant from the plane passing through QR, QS, &c. *h* and the circle QXRSTV drawn from the center Q is equal to the circle EPLMNO; and RSTVX is an equilateral pentagone. But EF, EG, EH, &c. and QX, QR, QS, &c. being conceived to be drawn; then because FRq *k* = FLq + LRq, *l* or EFq *m* = FGq. *n* therefore FR, FG, and so all RS, FG, FR, RG, GS, GH, &c. shall be equal one to the other, and consequently the ten triangles RFX, RFG, RGS, &c. are equilateral and equal. Moreover, because XQY is a *o* right angle; therefore XYq *p* = QXq + QYq *q* = VXq or FGq. wherefore XY, VX, and likewise YV, YT, YS, YR, ZG, ZH, &c. are equal. Therefore other ten triangles are made, equilateral and equal both

d *constr.*
e 6. 11.
f 33. 1.
g 15. 11.
h 1. *def.* 3.
k 47. 1.
l *constr.*
m 10. 13.
n *sch.* 48. 1. *and* 1. *ax.*
o *cor.* 14. 11.
p 47. 1.
q 10. 13.

both to one another, and to the ten former; and so an Icosaedron is made.

Moreover, divide equally EQ in α, draw the right
r 15. def. 1. lines, αF, αX, αV; and because QX r = QV, and
αQ the common side, and EQX, EQV are right an-
s 4. 1. gles, s therefore shall αX be = αV; and for the same
reason all the lines αX, αR, αS, αT, αV, αF, αG,
t 9. 13. αH, αI, αK are equal. But because ZQ. QE t :: QE.
u 3. 13. ZE. therefore Zαq u = 5 Eαq x = EQq (EFq) + Eαq
x 4. 2. y = αFq. therefore Zα = αF. in like manner αF = Yα.
y 47. 1. therefore the sphere, whose center is α and αF the radius, shall pass through the 12 angular points of the Icosaedron.

z 15. 5. Lastly, z because Zα, αE :: ZY. QE; a and so Zαq,
a 22. 6. αEq :: ZYq. QEq. b therefore ZYq = 5 QEq, or 5 BDq:
b 14. 5. but ABq. BDq c :: AB. BC :: 5. 1. d therefore ZY =
c cor. 8. 6. AB. *Which was to be done.* Therefore if AB be put $\acute{\rho}$,
d 1. ax. 1. e then EF = $\sqrt{AB \times BC}$ shall be also $\acute{\rho}$. and consequent-
e sch. 12. ly FG the side of the pentagone, and likewise of the Ico-
10. saedron, f is a minor line. *Which was to be demonstrated.*
f 11. 13.

Coroll.

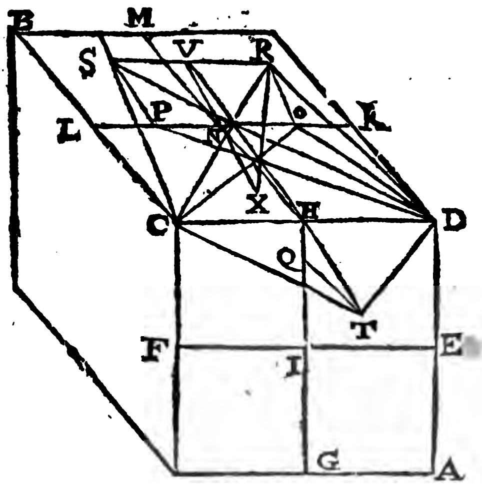
1. From hence is inferred, that the diameter of the sphere is in power quintuple of the semidiameter of the circle encompassing the five sides of the Icosaedron.

2. Also it is manifest that the diameter of the sphere is composed of the side of a hexagone, that is, of the semidiameter, and two sides of the decagone of a circle encompassing the five sides of the Icosaedron.

3. It appears likewise that the opposite sides of an Ico-
a 33. 1. saedron, such as RX, HI, are parallels. For RX a is
b sch. 26. parallel to LP. b parallel to HI.
3.

PROP.

PROP. XVII.



To describe a Dodecaedron, and comprehend it in the sphere wherein the former figures were comprehended; and to demonstrate that the side RS of the Dodecaedron is an irrational line of that sort which is called an apotome or residual line.

Let AB be a cube inscribed in the given sphere, and
let all the sides thereof be divided equally in the points
E, H, F, G, K, L, &c. and join the right lines KL, MH,
HG, EF. *a* make HI. IQ :: IQ. QH; and take NO, a 30. 6.
NP, = IQ. then erect OR, PS, perpendicular to the
plane DB, and QT to the plane AC; and let OR, PS,
QT, be equal to IQ, NO, NP. whence DR, RS, SC,
CT, TD, being connected, DRSCT shall be a pentagone
of the Dodecaedron required. For draw NV parallel to
OR, and having drawn NV out as far as the center of the a 47. 1.
cube X, join the right lines DS, [illegible] DP, CR, CP, HV, b 7. *ax.* 1.
HT, RX. Because DOq *a* = DKq (*b* KNq) + KOq *c* = 3 c 4. 13.
ONq (3 ORq) *d* thence DRq = 4 ORq *e* = OPq, or, RSq. d 47. 1.
therefore DR = RS. By a like way of reasoning DR, e 4. 2.
RS, SC, CT, TP are equal. But because OR *f* is = and f *const.* 96.
g parallel to PS, therefore RS, OP, and consequently RS, 11.
DC shall be also parallels. *h* therefore these with those g 33. 1.
that conjoin them DR, CS, VH, are in one and the same h 9. 1.
plane.

k 7. 11. plane. Moreover, because HI, IQ *k* :: IQ. (TQ.) QH
k *constr.* *k* :: HN. NV. and both TQ, HN, and QH, NV *k* are
l 6. 11. perpendicular to the same plane, *l* and so also parallels,
m 32. 6. *m* THV shall be a right line. *n* therefore the Trapezium
n 1, *and* 2. DRSC, and the triangle DTS are in one plane extended
11. through the right lines DC, TV. *o* therefore DTCSR is
o 5. 13. a pentagone, and that also equilateral, by what is shewn
p 47. 1. already. Furthermore, because PK. KN :: KN. NP; and
q 1. *ax.* 2. DSq *p* $=$ DPq $+$ PSq (PNq) $=$ *p* DKq $+$ PKq $+$ NPq, *q*
and 4. 13. thence DSq $=$ DKq $+$ 3 KNq $=$ 4 DKq (4 DHq) *r* $=$ DCq.
r 4. 2. therefore DS $=$ DC. whence the triangles DRS, DCT,
s 8. 1. are equilateral one to another. *s* therefore the angle
DRS $=$ DTC, and likewise the angle CSR $=$ DCT.
* 7. 13. therefore the * pentagone DTCSR is also equiangular.
Moreover, because AX, DX, CX, &c. are semidiameters
t 15. 13. of the cube, *t* thence is XN $=$ IH, or KN, *u* and so
u 1. *ax.* 1. XV $=$ KP; wherefore because RVX, is a *x* right angle,
x 29. 1. *z* thence RXq $=$ XVq $+$ RVq (NPq) $=$ KPq $+$ NPq
z 47. 1. *a* $=$ 3 KNq *b* $=$ AXq or DXq, &c. therefore RX, AX,
a 4. 13. DX, and for the same reason XS, XT, AX, are equal
b 15. 13. one to another. And if by the same method whereby the pentagone DTCSR was made, twelve like pentagones, touching the twelve sides of the cube, be made, they shall compose a Dodecaedron; and a sphere passing through their angular points, whose radius is AX, or RX, shall comprehend that Dodecaedron. *Which was to be done.*

c *constr.* Lastly, because KN. NO *c* :: NO. OK. *d* thence KL.
d 15. 5. OP :: OP. OK $+$ PL. Therefore if AB the diameter of
e 15. 13. the sphere be supposed $\dot{\rho}$, then shall KL *e* $= \sqrt{\frac{AB}{3}}$ *f* be
f *schol.* 12. 10. also $\dot{\rho}$. *g* whence OP, or RS the side of the Dodecaedron
g 6. 13. shall be a residual line. *Which was to be demonstrated.*

Coroll.

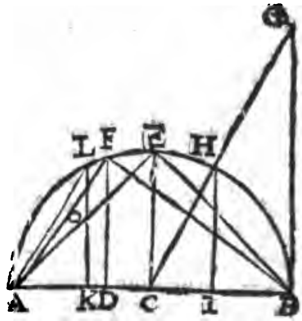
From this demonstration it follows, 1. That if the side of a cube be cut in extreme and mean proportion, the greater segment shall be the side of the Dodecaedron described in the same sphere.

2. If the lesser segment of a right line, cut in extreme and mean proportion, be the side of the Dodecaedron, the greater segment shall be the side of the cube inscribed in the same sphere.

3. It is manifest also, that the side of the cube is equal to the right line which subtendeth the angle of a pentagone of the Dodecaedron inscribed in the same sphere.

PROP.

PROP. XVIII.



To find out the sides of the five precedent figures, and compare them together.

Let AB be the diameter of the sphere given, and AEB the semicircle, and let AC be a 10. 1.
a = ½ AB, and AD b = ⅓ b 10. 6.
AB. then erect the perpendiculars CE, DF, and BG = AB. join AF, AE, BE, BF, CG; and let fall the perpendicular HI from H, and CK being taken equal to CI, from K erect the perpendicular KL; and join AL. Lastly, c make AF. c 30. 6.
AO : : AO. OF.

Therefore 3. 2 d : : AB. BD e : : ABq. BFq the side of d *constr.*
a Tetraedron. and 2. 1 : : a AB. AC : : ABq. BEq f the e *cor.* 8. 6.
side of an Octaedron. f 14. 13.

Also 3. 1 d : : AB. AD e : : ABq. AFq. g the side of g 15. 13.
an Hexaedron. h *constr.*

Moreover, because AF. AO h : : AO. OF. k thence k *cor.* 17.
shall AO be the side of a Dodecaedron. Lastly, BG, 13.
(2 BC.) BC l : : HI. IC. m therefore HI = 2 CI n = KI. l 4. 6.
therefore HIq o = 4 CIq. consequently CHq p = 5 CIq m 14. 5.
q therefore ABq = 5 KIq. r therefore KI, or HI is a radi- n *constr.*
us of a circle enclosing the pentagone of an Icosaedron; o 4. 2.
and AK or IB r is the side of a decagone inscribed in the p 47. 1.
same circle. s whence AL shall be the side of a penta- q 15. 5.
gone, t and also the side of an Icosaedron. Whereby it r *cor.* 16.
appears that BF, BE, AF are ρ̊ ⊐. and AL, AO ρ̊ ⊐, 13.
and BF ⊏ BE, and BE ⊏ AF, and AF ⊏ AO. And be- s 10. 13.
cause 3 AFq = ABq u = 5 KLq, and AF × AO ⊏ AF × t 16. 13.
OF, x and so AF × AO + AF × OF ⊏ 2 AF × OF, y that u 1. 6.
is, AFq ⊏ z 2 AOq. a thence shall AFq (5 KLq) be x 4. *ax.* 1.
⊏ 6 AOq. consequently KL ⊏ AO y and much rather y 1. 2.
AL ⊏ AO. z 17. 6.

That we may express these sides in numbers; If AB a 47. 1.
be supposed $\surd$. 60, then, reducing what is already shewn to supputation, BF = $\surd 40$, and BE = $\surd 30$, and AF = $\surd 20$. Also AL = $\surd 30 - \surd 180$ (for AK = $\surd 15 - \surd 3$. and KL (HI) = $\surd 12$.) Lastly, AO = $\surd : 30 - \surd 500$ ($\surd 25 - \surd 5$.)

Schol.

Schol.

It is very apparent that besides the five aforesaid figures, there cannot be described any other regular solid figure (viz. *such as may be contained under ordinate and equal plane figures.*)

For three plane angles at least are required to the constituting of a solid angle; *a* all which must be less than four right angles. *b* But 6 angles of an equilateral triangle, 4 of a square, and 6 of a hexagon, do severally equal 4 right angles; and 4 of a pentagon, 3 of a heptagon, 3 of an octagone, &c. do exceed 4 right angles: Therefore only of 3, 4, or 5 equilateral triangles, of 3 squares, or 3 pentagones, it is possible to make a solid angle. Wherefore besides the five above mentioned, there cannot be any other regular bodies.

a 21. 11. b See sch. 32. 1.

Out of P. Herigon.

The Proportions of the sphere and the five regular figures inscribed in the same.

Let the diameter of the sphere be 2, then shall the peripherie or circumference of the greater circle, be 6. 28318.
The superficies of the greater circle, 3. 14159.
The superficies of the sphere, 12. 56637.
The solidity of the sphere, 4. 1879.

The side of the tetraedron, 1. 62299.
The superficies of the tetraedron, 4. 6188.
The solidity of the tetraedron, 0. 15132.

The side of the hexaedron, 1. 1547.
The superficies of the hexaedron, 8.
The solidity of the hexaedron, 1, 5396.

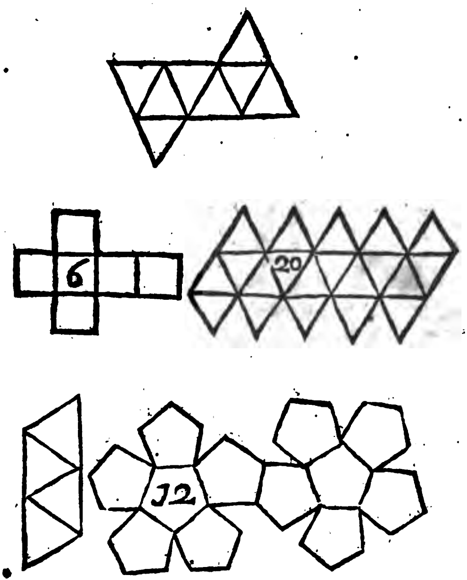
The side of the octaedron, 1. 41421.
The superficies of the octaedron, 6. 9282.
The solidity of the octaedron, 1. 33333.

The side of the dodecaedron, 0. 71364.
The superficies of the dodecaedron, 10. 51462.
The solidity of the dodecaedron, 2. 78516.

The

The side of the Icosaedron, 1. 05146.
The superficies of the Icosaedron, 9. 57454.
The solidity of the Icosaedron, 2. 53615.

If five equilateral and equiangular figures, like these in the schemes beneath, be made of Paper, and rightly folded, they will represent the five regular Bodies.

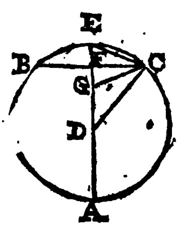


The End of the Thirteenth Book.

THE

THE FOURTEENTH BOOK OF *EUCLIDE's* ELEMENTS.

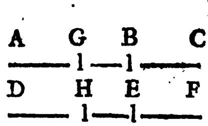
PROPOSITION I.



A Perpendicular line DF drawn from D the center of a circle ABC to BC the side of a pentagone inscribed in the said circle, is the half of these two lines taken together, viz. *of the side of the hexagone DE, and the side of the decagone EC inscribed in the same circle ABC.*

a 4. 1. Take FG = FE, and draw CG: *a*
b 5. 1. Then CE is = CG. therefore the angle CGE *b* = CEG
c 32. 1. *b* = ECD. therefore the angle ECG *c* = EDC. *d* = $\frac{1}{4}$
d *hyp. and* ADC *e* = $\frac{1}{2}$ CED ($\frac{1}{2}$ ECD.) *f* consequently the angle
33. 6. GCD = ECG = EDC. *g* wherefore DG = GC (CE.)
e 20. 3. therefore DF = CE (DG) + EF = $\frac{DE + CE.}{2}$ *Which*
f 7. *ax.* *was to be demonstrated.*
g 6. 1.

PROP. II.

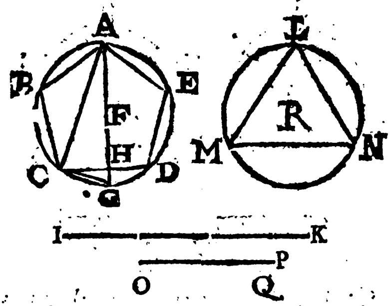


If two right lines AB, DE, are cut according to extreme and mean proportion (AB. AG :: AG. GB. and DE. DH :: DH. HE.) they shall be cut after the same manner, viz. *in the*
a 17. 6. *same proportions (AG. GB :: DH. HE.)*
b 8. 2. Take BC = BG; and EF = EH. Then AB × BG is
c 1. *ax.* 1. *a* = AGq. wherefore ACq *b* = 4ABG + AGq *c* = 5AGq.
d 22. 5. In like manner shall DFq = 5DHq. *d* therefore AC.
and 22. 6.

AG : : DF. DH. whence by compounding AC+AG. AG : : DF+DH. DH. that is, 2 AB. AG : : 2 DE. DH. e consequently AB. AG : : DE. DH; f whence by division AG. GB : : DH. HE. *Which was to be demonstrated.*

e 22. 5.
f 17. 5.

PROP. III.



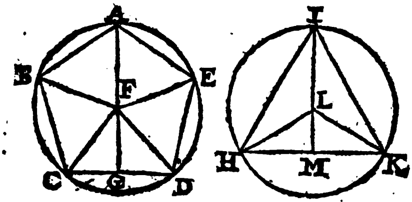
The same circle ABD comprehends both ABCDE the pentagone of a Dodecaedron, and LMN the triangle of an Icosaedron inscribed in the same sphere.

Draw the diameter AG, and the right lines AC, CG. and let IK be the diameter of the sphere, *a* and IKq = 5 OPq. *b* and make OP. OQ : : OQ. QP. Because ACq+CGq *c*=AGq *d* =4FGQ; and ABq *e*=FGq+CGq. *f* thence ACq+ABq=5 FGq. moreover, because CA. AB *g* : : AB. CA—AB; and OP. OQ : : OQ. QP. *h* and so CA. OP : : AB. OQ, *k* therefore 3 ACq (*l* IKq.) 5 OPq (*m* IKq) : : 3 ABq. 5 OQq. therefore 3 ABq= 5 OQq. But because ML *n* is the side of a pentagone inscribed in a circle, whose radius is OP, thence 15 RMq. *o*=5 MLq *p*=5 OPq+5 OQq=* 3 ACq+ 3 ABq *q*=15 FGq. *r* therefore RM=FG. *s* and consequently the circle ABD is=to the circle LMN. *Which was to be demonstrated.*

a *sch.* 47. 1.
b 30. 6.
c 47. 1.
d 4. 2.
e 10. 13.
f 2, *and* 3. *ax.*
g 8. 13.
h 2. 13. & 16. 5.
k 22. 6. & 4. 5.
l 15. 13.
m *constr.*
n *cor.* 16. 13.
o 12. 13.
p 10. 13.
q 15. 5. *and above.*
* *before.*
r 1. *ax.* 1. *and schol.* 48. 1.
s 1. *def.* 3

 PROP.

PROP. IV.



If from F the center of a circle encompassing the pentagone of a Dodecaedron ABCDE, a perpendicular line FG be drawn to one side of the Pentagone CD; the rectangle contained under the said side CD and the perpendicular FG, being thirty times taken, is equal to the superficies of a Dodecaedron. Also,

If from the center L of a circle inclosing the triangle of an Icosaedron HIK, a perpendicular line LM be drawn to one side of the triangle HK, the rectangle contained under the said side HK and the perpendicular LM, being thirty times taken, shall be equal to the superficies of an Icosaedron.

a 8. 1. Draw FA, FB, FC, FD, FE. *a* then shall the triangles CFD, DFE, EFA, AFB, BFC be equal, but CD×
b 41. 1. FG *b* = 2 triangles CFD. therefore 30 CD×GF *c* =
c 15. 5. 60 CFD*d* = 12 pentagones ABCDE *e* = to the superfi-
d 6. ax. cies of a Dodecaedron. *Which was to be demonstrated.*
e 17. 3. Draw LI, LH, LK; then HK×LM *f* is = 2 triangles
f 41. 1. LHK. therefore 30 HK×LM *g* = 60 HLK = 20 HIK *h*
g 15. 5. = to the superficies of an Icosaedron. *Which was to be*
h 16. 13. *demonstrated.*

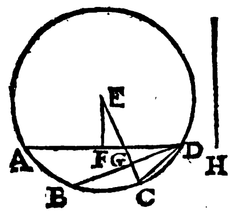
Coroll.

k 15. 5. CD×FG. HK×LM *k* : : the superficies of a Dodecaedron to the superficies of an Icosaedron.

PROP.

PROP. V.

The superficies of a Dodecaedron hath to the superficies of an Icosaedron inscrib'd in the same sphere, the same proportion that H the side of a cube hath to AD the side of an Icosaedron.

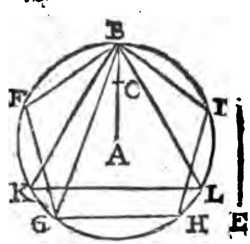


Let the circle ABCD *a* enclose both the Pentagone of a Dodecaedron, and the triangle of an Icosaedron; whose sides are BD, AD. upon which from the center E let fall the perpendiculars EF, EGC; and draw CD. a 3. 14.

Because EC+CD, EC *b* :: EC. CD. thence EG. b 9. 13.
(*c* ½ EC×½ CD.) EF (*d* ½ EC) *e* :: EF. EG—EF. c 1. 14.
(½ CD.) but H. BD *f* :: BD. H—BD. *g* therefore H. d *cor.* 12,
BD :: EG. EF. consequently H × EF = BD × EG. 13.
wherefore since H. AD *h* :: H×EF. AD×EF. *k* thence e 15. 5.
shall be H. AD :: BD×EG. AD×EF *l* :: the super- f *cor.* 17.
ficies of a Dodecaedron to the superficies of an Icosae- 13.
dron. *Which was to be demonstrated.* g 2. 14.
h 1. 6.
k 7. 5.
l *cor.* 4. 14.

PROP. VI.

If a right line AB be cut in extreme and mean proportion, then as the right line BF, containing in power that which is made of the whole line AB, and that which is made of the greater segment AC, is to the right line E containing in power that which is made of the whole line AB, and that which is made of the lesser segment BC; so is the side of the cube BG to the side of an Icosaedron BK inscribed in the same sphere with the cube.



In the circle, whose semidiameter is AB, inscribe BFGHI the pentagone of a Dodecaedron, and BKL the triangle of an Icosaedron, *a* wherefore BG shall be the side of a cube inscribed in the same sphere; therefore BKq *b* = 3 ABq; and Eq *c* = 3 ACq. therefore BKq. Eq. *d* :: ABq. ACq *e* :: BGq. BFq. wherefore by permutation BGq. BKq :: BFq. Eq. *f*. whence BG. BK :: BF. E. *Which was to be demonstrated.*

a *cor.* 17. 13.
b 12. 13.
c 4. 13.
d 15. 5.
e 2. 14.
f 22. 6.

PROP. VII.

A Dodecaedron is to an Icosaedron, as the side of a Cube is to the side of an Icosaedron, inscribed in one and the same sphere.

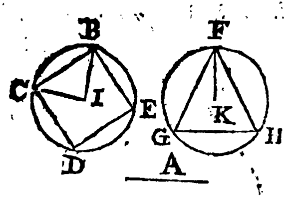
a 3. 14. Because *a* the same circle comprehends both the pentagone of a Dodecaedron. and the triangle of an Icosaedron, *b* the perpendiculars drawn from the center of the

b 47. 1. sphere to the planes of the pentagone and triangle, shall be equal one to another. Therefore if the Dodecaedron and Icosaedron be conceived divided into pyramids, right lines being drawn from the center of the sphere to all the angles, the altitudes of all the pyramids shall be equal one to the other. Wherefore since the pyra-

c 5. and 6. 12. mids are *c* of equal height with the bases, and the superficies of the Dodecaedron is equal to twelve pentagones; and the superficies of the Icosaedron to twenty triangles, the Dodecaedron shall be to the Icosaedron, as the superficies of the Dodecaedron is to the superficies of

d 5. 14. the Icosaedron, *d* that is, as the side of the cube is to the side of the Icosaedron.

PROP. VIII.



The same circle BCDB comprehends both the square of the cube BCDE; and the triangle of the octaedron FGH inscribed in one and the same sphere.

Let A be the diame-

a 15. 13. ter of the sphere. Because Aq *a*= 3 BCq *b*= 6 BIq;

b 47. 1. and also Aq *c* = 2 GFq; *d*= 6 KFq; thence shall BI

c 14. 13. be=KF. *e* therefore the circle CBED=GFH. *Which was*

d 12. 13. *to be demonstrated.*

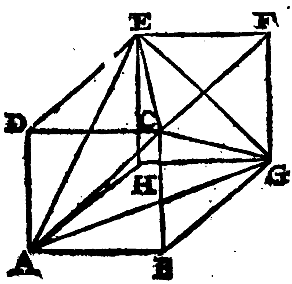
e 1. def. 3.

The End of the Fourteenth Book.

THE

THE FIFTEENTH BOOK OF *EUCLIDE*'s ELEMENTS.

PROPOSITION I.



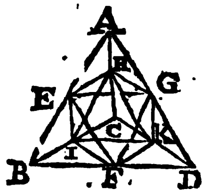
IN *a cube given ABGHDCFE to describe a pyramid AGEC.*

From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA. All which are *a* equal among themselves, as being the diameters of equal squares: therefore the triangles CAG, CGE, CEA, EAG are equilateral and equal; and consequently AGEC is a pyramid, which insists upon the angles of the cube, and therefore *b* is inscribed in it. *Which was to be done.*

a 47. 1.

b 31. *def.* 11.

PROP. II.

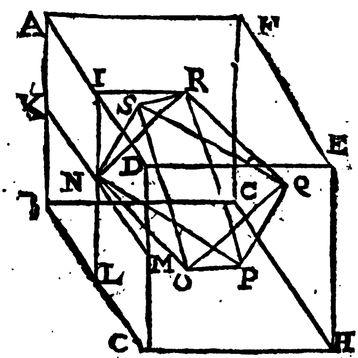


In a pyramid given ABDC to describe an Octaedron EGKIFH.

a 10. 1. *a* Bisect the sides of the pyramid in the points, E, I, F, K, G, H, which join with the 12 right lines EF, FG, GE, &c. All these are *b* equal one to the other; consequently the 8 triangles EHI, IHK, &c. are equilateral and equal, and so make *c* an Octaedron described *d* in the given pyramid. *Which was to be done.*

b 4. 1.
c. 27. *def.* 11.
d 31. *def.* 11.

PROP. III.



In a cube given CHGBDEFA to describe an Octaedron NPQSOR.

* 8. 4. Connect * the centers of the squares N, P, Q, S, O, R with the twelve right lines NP, PQ, QS, &c. which are *a* equal among themselves; and so make 8 equilateral and equal triangles: wherefore *b* the Octaedron NPQSOR *b* is inscribed in the cube. *Which was to be done.*

a 4. 1.
b 31. *and* 27. *def.* 11.

PROP.

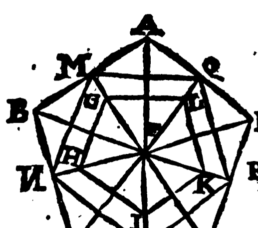
PROP. IV.

In an Octaedron given ABCDEF, to inscribe a cube.

Let the sides of the pyramid EABCD, whose base is the square ABCD, be bisected by the right lines, LM, MN, NO, OL, which are *a* equal and *b* parallel to the sides of the square ABCD. *c* then the quadrilateral LMNO is a square. In like manner, if the sides of the square LMNO be bisected in the points G, H, K, I, and GH, HK, KI, IG connected, GHKI shall be a square. And if in the other 5 pyramids of the Octaedron, the centers of the triangles be in the same sort conjoined with right lines, then other squares will be described like and equal to the square GHKI. wherefore six such squares shall make a cube, which shall be described within an Octaedron, *d* since its eight angles touch the eight bases of the Octaedron in their centers. *Which was to be done.*

a 4. 1.
b 2. 6.
c 29. *def.* 1.
d 31. *def.* 11.

PROP. V.



In an Icosaedron given to inscribe a Dodecaedron.

Let ABCDEF be a pyramid of the Icosaedron, whose base is the pentagone ABCDE; and the centers of the triangles G, H, I, K, L; which connect with the right lines GH, HI, IK, KL, LG. Then GHIKL shall be a pentagone of the Dodecaedron to be inscribed.

For the right lines, FM, FN, FO, FP, FQ, passing
a cor. 3. 3. through the centers of the triangles, *a* do bisect their ba-
b 4. 1. ses. *b* therefore the right lines MN, NO, OP, PQ, QM *c*
c 4. 1. are equal one to the other; *d* whence also the angles
d 8. 1. MFN, NFO, OFP, PFQ, QFM are equal; therefore
e 4. 1. the pentagone GHIKL is equiangular, *e* and consequently
f 12. 13. equilateral, since FG, FH, FI, FK, FL *f* are equal. And if in the other eleven pyramids of the Icosaedron, the centers of the triangles be in like sort conjoined with right lines, then will pentagones equal and like to the pentagone GHIKL be described. Wherefore 12 of such pentagones shall constitute a Dodecaedron; which also shall be described in the Icosaedron, seeing the twenty angles of the Dodecaedron consist upon the centers of the twenty bases of the Icosaedron. Whereby it appears that we have described a Dodecaedron in an Icosaedron given. *Which was to be done.*

FINIS.

DEFINITIONS.

I. PLanes or Spaces, Lines, and Angles, to which we can find others equal, are said to be given in Magnitude.

II. A Ratio is said to be given, when we can find it, or one equal to it.

III. Rectiline figures, whose angles are given, and also the ratio of the sides to one another, are said to be given in Species or Kind.

IV. Points, Lines and Angles, which have and keep always one and the same place and situation, are said to be given in Position or Situation.

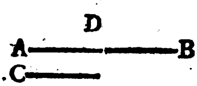
V. A Circle is said to be given in Magnitude, when the semidiameter thereof is given in Magnitude.

VI. A Circle is said to be given in Position, and Magnitude, when the Center thereof is given in Position, and the semidiameter in Magnitude.

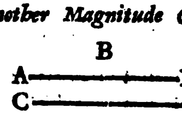
VII. Segments of Circles, whose angles and bases are given in Magnitude, are said to be given in Magnitude.

VIII. Segments of a Circle, whose angles are given in Magnitude, and the bases of the segments in Position and Magnitude, are said to be given in Position and Magnitude.

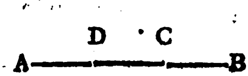
IX. A Magnitude AB, is greater than another Magnitude C, by a given Magnitude BD, when having taken away the given Magnitude DB, the rest AD, is equal to the other Magnitude C.



X. A Magnitude AB, is less than another Magnitude C, by a given Magnitude BD, when having added thereto the given Magnitude BD, the whole AD is equal to the other Magnitude C.



XI. A Magnitude AB, is said to be greater than another Magnitude CB, by a given Magnitude AD, and in ratio, when taking from the same Magnitude the given Magnitude AD, the rest DB, hath to the other Magnitude CB, a given ratio.



XII.

XII. *A Magnitude AB is said to be less than another Magnitude BC, by a given Magnitude AD, and in ratio, when the given Magnitude AD being added thereto, the whole DB hath to the other Magnitude BC, a given ratio.*

XIII. *A right line is said to be drawn down from a given point, unto a right line given in Position, the right line being drawn in a given angle.*

XIV. *A right line is said to be drawn up from a given point, to a right line given in Position, the right line being drawn in a given angle.*

XV. *A right line is against another right line in Position, when it is drawn parallel thereto through a given point.*

PROPOSITION I.

TWO *Magnitudes A and B being given, the ratio they have to one another A to B is also given.*

Demonstration. For seeing that the Magnitude A is given,
a 1. *def.* *a* we can find one equal thereto, which let be C. Again, forasmuch as the Magnitude B is given, we can also find one equal to that, and let that be D. Therefore seeing that A is
b 7. 5. equal to C, and B to D, as A is to C, *b* so is B to D,
c 16. 5. and by permutation, *c* as A shall be to B, so C shall be
d 2. *def.* to D. Therefore *d* the ratio of A to B is given, for it is the same ratio as of C to D, as we have found, and which ought to be demonstrated.

PROP. II.

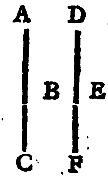
If a given Magnitude A, hath to some other Magnitude B, a given ratio, that other Magnitude B, is also given in Magnitude.

Demonst. For seeing that A is given, we can find one equal thereto, which let be C: And forasmuch as the ratio of A to B, is also given, we can find
a 2. *def.* *a* one of the same. Let it be found, and let the ratio be of C to D. Now seeing that as A is to B, so C is to D; and by permutation, as A is to C, so B is to D: But A is equal to C, therefore *b* B
b 14. 5. shall be also equal to D. Therefore *c* the Magnitude B is
c 1. *def.* given, seeing that thereto there hath been found one equal, to wit, D.

PROP.

PROP. III.

If given Magnitudes AB and BC, are compounded, that Magnitude AC, that is compounded of them, shall be also given.



Demonstr. For seeing that AB is given, we can find one equal to it, which let be DE. Again, seeing that BC is given we can also find one equal to that, which let be EF. Wherefore seeing that DE is equal to AB, and EF is equal to BC, the whole AC *a* is equal to the whole DF. Therefore AC is given, seeing that DF is proposed equal thereto. a 2. *ax.* 1.

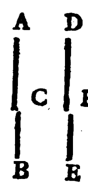
PROP. IV.

If from a given Magnitude AB, there be taken away a given Magnitude AC, the remaining Magnitude CB is also given.

Demonstr. Forasmuch as AB is given, we can find one equal thereto, which let be DE. Again, seeing that AC is given, we can also find one equal to it, which let be DF. Seeing then that the Magnitude AB is equal to the Magnitude DE, and the Magnitude AC to the Magnitude DF; the rest CB *a* shall be equal to the rest FE. Wherefore CB is given, for to it there hath been found an equal, to wit, FE. a 3. *ax.* 1.

PROP. V.

If a Magnitude AB, hath a given ratio to some part thereof AC, it will have also a given ratio to the part remaining CB.



Demonstr. Let DE be exposed as a given Magnitude, and seeing that the ratio of the Magnitude AB, to the Magnitude AC, is given, *a* we can find one of the same, which let be DE to DF; therefore the ratio of the same DE to DF is given. But DE being given, so is *b* also its part DF; and consequently, *c* the rest FE: Therefore *d* seeing that DE and FE are given, the ratio of the same DE to FE is also given. And forasmuch as DE is to DF, as AB is to AC, and by conversion, as DE to FE, so AB is to CB. But the ratio of DE to FE, is given, as hath been demonstrated; therefore the ratio of AB to CB is also given.

a 2. *def.*
b 2. *prop.*
c 4. *prop.*
d 1. *prop.*
19. 5.

Scholium.

Scholium.

From this it is evident that if a Magnitude hath to some part thereof a given ratio, by division, the ratio that one part hath to the other, shall be also given. For seeing that as DE is to FE, so is AB to CB; by division, as DF to FE, so AC to CB. But it hath been demonstrated that the parts DF and EF are given, and consequently their ratio is also given: In like manner, therefore, the ratio of AC to CB is given.

PROP. VI.

If two Magnitudes AB and BC, having to one another a given ratio, are compounded, the Magnitude AC compounded of them, shall also have a given ratio to each of them AB and BC.

Demonst. Let the given Magnitude DE be exposed, and seeing that the reason of AB to BC is given; let there be made one and the same of the said DE to EF; therefore the ratio
a 2. *prop.* of the same DE to EF is given; and therefore *a* the Magnitude DE being given, both the one and the other
b 3. *prop.* of them DE and FE, is given. Wherefore *b* the whole
c 1. *prop.* DF shall be also given. Therefore *c* the ratio of the same DF to each of them DE and EF, shall be given. And forasmuch then as AB is to BC, so is DE to EF;
d 18. 5. by compounding, *d* as AC is to BC, so is DF to EF: Therefore by conversion, as AC to AB, so is DF to DE. Therefore as the whole DF is to each of the other Magnitudes DE and FE, so the whole AC is to each of
e 2. *def.* the Magnitudes AB and BC. Therefore *e* the ratio of the same AC to each of the Magnitudes AB and BC is given.

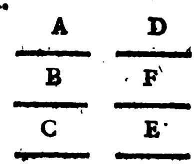
PROP. VII.

If a given Magnitude AB be divided according to a given ratio AC to CB, each segment AC and CB is given.

Demonstr. For seeing the ratio of AC to CB is given,
a 6. *prop.* the ratio of *a* AB to each of them (AC and CB) is
b 2. *prop.* also given. But AB is given: Therefore *b* each of the segments AC and CB is also given.

PROP.

PROP. VIII.

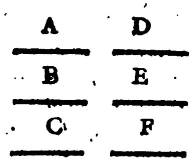


Magnitudes A and C, which have to one and the same a given ratio B, shall be to one another in a given ratio A to C.

Demonstr. For let the given magnitude D be exposed, and seeing that the ratio of A to B is given, let the same be done of the said D to E. Now seeing that D is given, *a* E is also given. Again, seeing that the ratio of B to C is given, let the same be done of E to F. But E is given, and therefore F is also given. But seeing that D is given, *b* the ratio of the same D to F is given; and seeing that as A to B, so D to E, and as B to C, so is E to F; in ratio of equality, *c* as A is to C, so is D to F; but the ratio of D to F is given. Therefore the ratio of A to C is also given.

a 2. *prop.*
b 1. *prop.*
c 22. 5.

PROP. IX.



If two or more magnitudes A, B, and C, are to one another in a given ratio, and that the same magnitudes A, B, and C, have to other magnitudes D, E, and F, given ratio's, although they be not the same, those other magnitudes D, E, and F shall be also to one another in given ratio's.

Demonstr. Forasmuch as the ratio of A to B is given, as also that of A to D, the ratio of D to B shall be given; But the ratio of B to E is also given; therefore the ratio of the same D to E shall be in like manner given. Again, seeing that the ratio of B to C is given, and also that of B to E, the ratio of E to C shall be given. But the ratio of C to F is also given. Therefore *a* the ratio of E to F shall be given. But it hath been demonstrated that the ratio of D to E is also given; and therefore *b* the ratio of D to F shall be given. Therefore the magnitudes D, E, and F are to one another in given ratio's.

a 8. *prop.*
b 8. *prop.*

PROP. X.

If a magnitude AB, be greater than another magnitude BC, by a given magnitude, and in ratio, the magnitude AC compounded of

of both, shall be also greater than that same magnitude, by a given magnitude, and in ratio; but if that compounded magnitude be greater than the same magnitude, by a given magnitude, and in ratio; either the remainder shall be also greater than that same, by a given magnitude, and in ratio; or else the same remainder is given with the following, to which the other magnitude hath a given ratio.

Demonstr. For seeing that AB is greater than BC by a given magnitude, and in ratio, let the given magnitude AD be taken away. Therefore *a* the reason of the remainder DB to BC is given; and by compounding, *b* the ratio of DC to BC is also given. But the magnitude AD is also given; therefore AC is greater than the same BC by a given magnitude, and in ratio.

a 11. *def.*
b 6. *prop.*

D B E
A——————C

Again, Let the magnitude AC be greater than the magnitude BC, by a given magnitude, and in ratio: I say, that the rest AB, is either greater than the same BC by a given magnitude, and in ratio; or that the same AB, with that which followeth, to which BC hath a given ratio, is given.

Forasmuch as the magnitude AC is greater than the magnitude BC, by a given magnitude, and in ratio, cut off from it the given magnitude: Now the same given magnitude is either less than the magnitude AB, or greater: Let it in the first place be less, and let it be AD. Therefore the ratio of the remainder DC to CB is given. Wherefore by division, the ratio of DB to BC is given. But the magnitude AD is also given; therefore the magnitude AB is greater *c* than the magnitude BC by a given magnitude, and in ratio. Now let the given magnitude be greater than the magnitude AB, and let AE be put equal thereto; therefore *d* the ratio of the remainder EC to CB is given; and by conversion, *e* the ratio of the same BC to BE, is also given. But the same EB with BA is given, for that the whole AE is given: Therefore there is given AB, with that which follows BE, to which BC hath a given ratio.

c 11. *def.*
d 11. *def.*
e 5. *prop.*

PROP.

PROP. XI.

If a magnitude AB be greater than a magnitude BC, by a given magnitude, and in ratio, the same magnitude AB, shall be also greater than the magnitude compounded of them by a given magnitude, and in ratio, and if the same magnitude be greater than the two others together by a given magnitude, and in ratio, that same magnitude shall be also greater than the rest by a given magnitude, and in ratio.

Demonstr. For seeing that the magnitude AB is greater than BC by a given magnitude and in ratio; let there be taken from it a given magnitude AD: Therefore *a* the ratio of the rest DB to DC, is given, and a 11. *def.*
therefore *b* the ratio of DC to BD shall be also given: b 6. *prop.*
Let the same be done of AD to DE, therefore the ratio of the same AD to DE is given. But AD is given, therefore *c* DE is also given, and consequently, *d* the rest c 2. *prop.*
AE, is also given. But seeing that as AD is to DE, so d 4. *prop.*
is DC to BD; by permutation, *e* as AD is to DC, so is e 16. 5.
DE to DB: Therefore by compounding, *f* as AC is to f 18. 5.
CD, so is EB to DB; and by permutation, *g* as AC is g 16. 5.
to EB, so is DC to DB. But the ratio of DC to DB is given: Therefore also is AC to EB, and consequently, that of EB to AC. But it hath been demonstrated that AE is given, therefore *h* AB is greater than AC by a h 11. *def.*
given Magnitude, and in ratio.

But now let AB be greater than AC by a given Magnitude, and in ratio: I say, that the same AB is also greater than the rest BC by a given Magnitude, and in ratio.

For seeing that AB is greater than AC by a given Magnitude, and in ratio. Let the given Magnitude AB be cut off therefrom: Therefore *i* the ratio of the i 11. *def.*
remainder EB to AC is given, and consequently also shall be given that of AC to EB. Let the same be done of AD to DE, therefore the ratio of AD to DE is given; and by conversion, *k* the ratio of AD to AE shall k 5. *prop.*
be also given, and consequently that of AE to AD. Now AE is given, therefore the whole AD *l* shall be also l 2. *prop.*
given; and seeing that as the whole AC is to the whole EB, so the part cut off AD, is to the part cut off ED, so also shall be *m* the remainder DC to the remainder DB. m 19. 5.
But the ratio of AC to EB is given: Therefore also

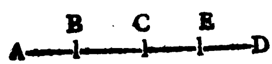
U shall

shall be given that of DC to DB. Wherefore by division,
n schol. 5. prop. *n* the ratio of BC to DB is given; and consequently also
shall be given that of DB to BC. But it hath been de-
o 11. def. monstrated that AD is given: Therefore *o* AB is greater
than the same BC by a given Magnitude, and in ratio.

PROP. XII.

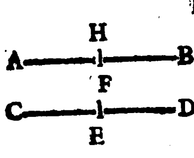
If there are three magnitudes AB, BC, and CD, and that the first AB, with the second BC, to wit, AC, be given. And the second BC, with the third CD, to wit, BD, be also given: Either the first AB shall be equal to the third CD, or the one shall be greater than the other by a given Magnitude.

Demonstr. Forasmuch as each of the Magnitudes AC and BD are given, the given Magnitudes are either equal to one another, or unequal. Let them be first equal: Therefore AC is equal to BD; take away the common
a 3. ax. 1. Magnitude BC, and there will remain *a* AB, equal to CD. But suppose them to be unequal as in this second figure, and let BD be greater than AC: Let then BE be put equal to AC:



Now seeing that AC is given, BE is also given. But the whole BD is also given,
b 4. prop. the rest ED *b* shall be so also; and forasmuch as BE is
c 3. ax. 1. equal to AC, taking away the common Magnitude *c* BC, there will remain AB equal to CE. But ED is given: Therefore CD is greater than AB by the given Magnitude ED.

PROP. XIII.

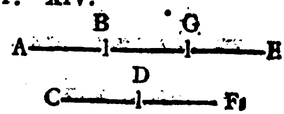


If there are three magnitudes AB, CD, and E, and that the first of them AB, hath a given ratio to the second CD; but the second CD is greater than the third E, by a given magnitude, and in ratio; also the first AB, shall be greater than the third E, by a given magnitude, and in ratio.

Demonstr. For seeing that CD is greater than E by a given Magnitude, and in ratio; let the given Magnitude CF be taken therefrom: Therefore the ratio of the Rest FD to E is given. And forasmuch as the ratio of AB to CD is given, let the same be done of AH to CF. Therefore the ratio of the same AH to CF is given.

given. But CF is given: Therefore *a* AH is also given; a 2 *prop.*
And seeing that as the whole AB is to the whole CD,
so the part cut off AH is to the part cut off CF, and so
b also the rest HB is to the rest FD, the ratio of the b 19. 5.
same HB to FD is also given. But the ratio of FD to
E is also given: Therefore *c* the ratio of HB to F is c 8. *prop.*
given. But it hath been demonstrated that AH is gi-
ven: Therefore *d* AB is greater than the said E by a gi- d 11. *def.*
ven Magnitude, and in ratio.

PROP. XIV.

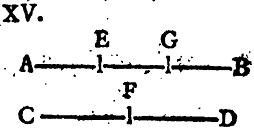


If two Magnitudes AB and CD, have to one another a given ratio, and that to each of them there be added a given Magnitude, to wit, BE and DF; either the whole AE and CF shall have to one another a given ratio, or the one shall be greater than the other by a given Magnitude, and in ratio.

Demonstr. For seeing that each of those Magnitudes
BE and DF, is given, *a* the ratio of the said BE
and DF is also given; and if that ratio be the same a 1. *prop.*
with that of AB to CD, that of the whole AE to the
whole CF, *b* shall be the same; and therefore the ratio
of the said AE to CF is given. b 12. 5.

Now let the ratio of BE to DF be not the same
with that of AB to CD, and let it be as AB to CD, so
BG to DF. Therefore the ratio of the said BG to
DF is given. But the Magnitude DF is given, there-
fore *c* BG is also given; and seeing that the whole BE c 2. *prop.*
is given, *d* the rest GE shall be also given. But forasmuch d 4. *prop.*
as AB is to CD, as BG is to DF, *e* so also is the whole e 12. 5.
AG to the whole CF; and therefore the ratio of the said
AG to CF is given: But the Magnitude GE is given:
Therefore *f* the Magnitude AE is greater than the Mag- f 11. *def.*
nitude CF by a given Magnitude, and in ratio.

PROP. XV.

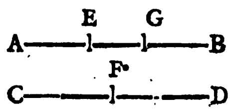


If two Magnitudes AB and CD, have to one another a given ratio, and that from each of them be taken away a given Magnitude (to wit, from the Magnitude AB the Magnitude AE, and from the Magnitude CD the Magnitude CF) the remaining Magnitudes EB and FD, either shall have to one another, a given ratio, or

or the one of them shall be greater than the other by a given Magnitude, and in ratio.

Demonstr. For seeing that each Magnitude AE and CF is given, the ratio of AE to CF is given; and if it be the same with that of AB to CD, that of the remainder

a 19. 5. EB to the remainder FD, *a* shall be also the same; and therefore the ratio of the said EB to FD shall be also given. But if it be not the same, let it be as AB to CD, so AC to CF. Now the ratio of AB to CD is given, therefore also that of AG to CF shall be given. But CF is given, there-



b 2. *prop.* fore *b* AG is given. But AE is also given, therefore *c*

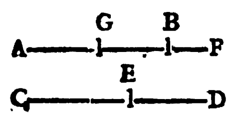
c 4. *prop.* the rest EG is given; and seeing that as AB is to CD, so the part cut off AG is to the part cut off CF, and so

d 4. *prop.* also is *d* the rest GB to the rest FD; the ratio of the said GB to FD is also given. Therefore seeing that EG

e 11. *def.* is given, EB is greater than FD *e* by a given Magnitude, and in ratio.

PROP. XVI.

If two Magnitudes AB and CD, have to one another a given ratio, and that from one of them, to wit, CD, there be taken away a given Magnitude DE, and to the other AB there be added a given Magnitude BF, the whole AF shall be greater than the rest CE, by a given Magnitude, and in ratio.



Demonstr. For seeing that the ratio of AB to CD is

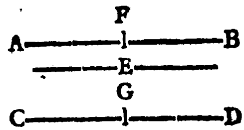
a 2. *def.* given, let the same be made of BG to DE: Therefore *a* the ratio of the said BG to DE is given. But DE is

b 2 *prop.* given therefore *b* BG is also given. But BF is also given,

c 3. *prop.* therefore *c* the whole GF is given. And seeing that as AB is to CD, so the part cut off BG, is to the part cut

d 19. 5. off DE; and *d* so also is the remainder AG to the remainder CE; the ratio of the said AG to CE is given: But GF is given, therefore the Magnitude AF is greater than the Magnitude CE by a given Magnitude, and in ratio.

PROP. XVII.

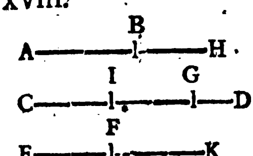


If there are three Magnitudes AB, E, and CD, and that the first AB be greater than the second E by a given Magnitude, and in ratio: And the third CD be also greater than the same second E, by a given Magnitude, and in ratio; the first AB shall have to the third CD, either a given ratio, or else the one shall be greater than the other by a given Magnitude, and in ratio.

Demonstr. For seeing that AB is greater than E by a given Magnitude, and in ratio, let the Magnitude AF be taken away: Therefore the ratio of the remainder FB to E is given. Again, seeing that CD is greater than the said E by a given Magnitude, and in ratio, let the given Magnitude CG be cut off therefrom; and the ratio of the remainder GD to E shall be given: Therefore *a* the ratio of FB to GD shall be also given. But to the said FB and GD are added the given Magnitudes AF and CG: Therefore the whole AB and CD *b* shall either have to one another a given ratio, or the one shall be greater than the other by a given Magnitude, and in ratio.

a 8. *prop.*
b 14. *prop.*

PROP. XVIII.



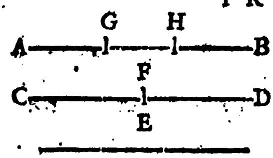
If there are three Magnitudes AB, CD, and EF, and that the one of them, to wit, CD, be greater than either of the other AB or BF, by a given Magnitude, and in ratio; either the two others AB and BF, shall have to one another a given ratio, or the one shall be greater than the other by a given Magnitude, and in ratio.

Demonstr. Forasmuch as the Magnitude CD is greater than the Magnitude AB by a given Magnitude, and in ratio, let the given Magnitude DG be taken therefrom: Therefore the ratio of the remainder CG to AB is given. Let the same be made of GD to BH, therefore the ratio of the said DG to BH is given. But DG is given, therefore *a* BH is also given. And seeing that as CG is to AB, so is GD to BH, *b* so also is the whole CD to the whole AH, the ratio of the said CD to AH shall be also given.

a 2. *prop.*
b 12. 5.

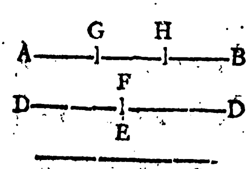
Again, seeing that the same CD is greater than EF by a given Magnitude, and in ratio; let the Magnitude DI be cut off therefrom: Therefore the ratio of the remainder CI to EF is given: Let the same be made of DI to FK. Therefore the reason of the said DI to FK shall be also given. But DI is given, therefore FK is also given. And seeing that as CI is to EF, so is ID to FK;
c 12. 5. so also is the whole c CD to the whole EK; the ratio of the said CD to FK shall be given. But the ratio
d 8. prop. of the same CD to AH is also given: Therefore d the ratio of the said AH to EK shall be given. And seeing that from the said AH and EK, the given Magnitudes
e 15. prop. BH and FK are cut off, the Magnitudes AB and EF e are either in a given ratio to one another, or the one is greater than the other by a given Magnitude, and in ratio.

PROP. XIX.



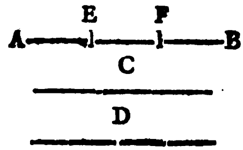
If there are three Magnitudes AB, CD, and E, and that the first AB be greater than the second CD, by a given Magnitude, and in ratio; and that the second CD be greater than the third E, by a given Magnitude, and in ratio; also the first Magnitude AB shall be greater than the third E, by a given Magnitude, and in ratio.

Demonstr. For seeing that CD is greater than E by a given Magnitude, and in ratio; let the given Magnitude, CF be taken therefrom: Therefore the ratio of the remainder FD to E is given. Again, seeing that AB is greater than the same CD by a given Magnitude, and in ratio: Let the Magnitude AG be taken therefrom: Therefore the ratio of the remainder GB to CD is given: Let the same be made of GH to CF. Therefore the ratio of the said GH to CF is given. But CF is given: Therefore also GH is given, and then AG is also
a 3. prop. given, the whole a AH shall be also given. But as GB is to CD, so is GH to CF, and
b 19. 5. so also b the remainder HB to the remainder FD: Therefore the ratio of the said HB to FD is given. But the ratio of the same FD to E is also given: Therefore the ratio of HB to E is in like manner



given, and so is also the Magnitude AE: Wherefore, the Magnitude AB *c* is greater than E by a given Magnitude, and in ratio. c 11. *def.*

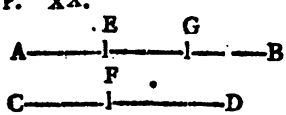
OTHERWISE.



Construction. Let there be three Magnitudes AB, C, and D, and let AB be greater than C by a given Magnitude, and in ratio; but let C be also greater than D, by a given Magnitude, and in ratio: I say, that AB is greater than D by a given Magnitude, and in ratio.

Demonstr. Forasmuch as AB is greater than C by a given Magnitude, and in ratio, let the given Magnitude AE be cut off therefrom: Therefore the ratio of the remainder EB to C is given. But the Magnitude C is greater than the Magnitude D by a given Magnitude, and in ratio; therefore *d* EB is greater than D by a given Magnitude, and in ratio: Wherefore let the given Magnitude EF be cut off therefrom; and the ratio of the remainder FB to D shall be given. But AF is *e* given. Therefore AB is greater than D by a given Magnitude, and in ratio. d 13. *prop.* e 3. *prop.*

PROP. XX.

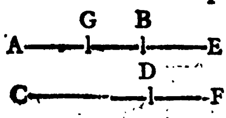


If there are two given Magnitudes, AB and CD, and that from them there are taken Magnitudes AE and CF, having to one another a given ratio; either the remaining Magnitudes EB and FD, shall have to one another given ratio's; or else the one shall be greater than the other by a given Magnitude, and in ratio.

Demonstr. For seeing that both the Magnitudes AB and CD, are given, the ratio of the said AB to CD is *a* also given; and if it be the same as of AE to CF, that of the remainder EB to the remainder FD shall be *b* also the same; and therefore the ratio of the said EB to FD shall be also given. But if it be not the same, let it be so as that AE be to CF, as AG to CD. Now the ratio of the said AE to CF is given: Therefore the ratio of the said AG to CD is given. But CD is given, therefore *c* AG a 1. *prop.* b 19. 5. c 3. *prop.*

is also given. But the whole AB is likewise given,
d 4. prop. therefore *d* the remainder BG is given. And seeing that as AE is to CF, so is AG to CD, and also the remainder EG to the remainder FD, the ratio of the said EG to FD is given. But GB is also given: Therefore the Mag-
e 11. def. nitude EB is greater *e* than the Magnitude BD by a given Magnitude, and in ratio.

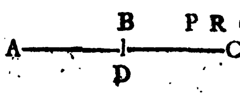
PROP. XXI.



If there are two Magnitudes given, AB and CD; and to them are added other Magnitudes BE and DF, having to one another a given ratio; either the whole AE and CF shall have to one another a given ratio, or else the one shall be greater than the other by a given Magnitude, and in ratio.

Demonstr. For seeing that both the Magnitudes AB
a 1. prop. and CD are given; their ratio *a* is also given; and if it be the same ratio as of BE to DF, the ratio of the whole AE to the whole CF shall be also given; for it
b 12. 5. shall be *b* the same. But if it be not the same, let it be as BE is to DF, so BG to CD: Therefore the ratio of the said BG to CD is given. But CD is given;
c 2. prop. therefore *c* also BG shall be given. But the whole AB is
d 4. prop. given; therefore also the *d* remainder AG shall be given. And seeing that as BE is to DF, so is BG to CD, and
e 12. 5. also *e* the whole GE to the whole CF, the ratio of the said GE to CF shall be likewise given. But AG is given; therefore the Magnitude AE is greater than the Magnitude CF by a given Magnitude, and in ratio.

PROP. XXII.

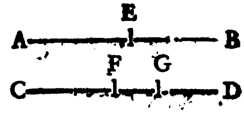


If two Magnitudes AB and BC, have to some other Magnitude D, a given ratio, also their compound Magnitude AC, shall have to the same Magnitude D, a given ratio.

Demonstr. For seeing that each Magnitude AB and
a 8. prop. BC hath a given ratio to D, the ratio *a* of AB to
b 6. prop. BC is given; and by compounding, *b* the ratio of AC to BC is given. But that of BC to D is also given,
c 8. prop. therefore *c* the ratio of the said AC to D shall be likewise given.

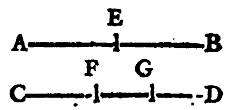
PROP.

PROP. XXIII.



If the whole AB be to the whole CD in a given ratio, and that the parts AE and EB be to the parts CF and FD in given ratio's, altho' they be not the same, the whole (to wit, AB, AE, and BE,) shall be to the whole (to wit, CD, CF, and FD,) in given ratio's.

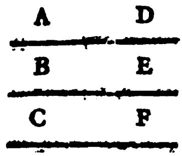
Demonstr. For seeing that AE is to CF in a given ratio, let the same be made of AB to CG; therefore the ratio of the said AB to CG is given; and consequently also that *a* of the rest EB to the rest FG. But the ratio a 19. 5.
of FD to the same EB is also given: Therefore the ratio of FD to FG *b* is likewise given, and therefore *c* that of b 8. *prop.*
FD to the remainder GD is also given. But the ratio c 5. *prop.*
of AB to each of the Magnitudes CD and CG is given; Therefore *d* also the ratio of CD to CG is given, and d 8. *prop.*
again *e* that of CD to the remainder GD. But the ra- e 5. *prop.*
tio of FD to DG is given, therefore also *f* that of the f 8. *prop.*
same CD to FD, and consequently that of *g* CD to the g 5. *prop.*
remainder FC; and therefore also the ratio of CF to FD shall be given. But the ratio of EB to FD is proposed to be given; therefore the ratio of CF to EB shall be given. Again, for that



the ratio of AB to CD is given; and also that of CD to each of those FC and FD, the ratio of the same AB to each of the said FC and *h* FD, shall be likewise given. h 8. *prop.*
But the ratio of the said FD to EB is given: Therefore the ratio of AB to BE shall be also given, and consequently AB to the remainder *i* AE. Wherefore by divi- i 5. *prop.*
sion *k* the ratio of AE to EB shall be likewise given. k *sch.* 5. *pr.*
But the ratio of EB to FD is given. Therefore also that of AE to FD. In like manner, seeing that the ratio of CD to AB is given; and that of AB to each of his parts AE and EB; also the ratio of the said CD to each of the said AE and EB, *l* shall be given: Where- l 8. *p. pro*
fore each of the Magnitudes AB, CD, AE, EB, CF, and FD, is to each of the others in a given ratio.

PROP.

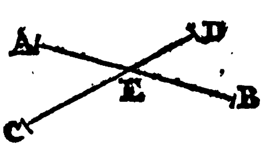
PROP. XXIV.



If of three right lines A, B, and C, proportional A to B, as B to C, the first A hath to the third C a given ratio, it will also have to the second B a given ratio.

Demonstr. For, let there be exposed another right line D, and seeing that the ratio of A to C is given; let the same be made of D to F; therefore the ratio of D to F is given. But D is given, therefore F is also given; betwixt the two
a 13. 7. right lines D and F, let there be taken *a* a mean proportional E. Therefore the rectangle made under D and F
b 17. 6. is equal *b* to the square of E. But the same rectangle of
c 3. *def.* D and F is *c* given: (for all the angles of that rectangle are given, being right angles, and the ratio's that the sides have to one another are also given;) therefore the square of E is given, and consequently the same right line E is also given (for one equal thereto may be found,
d 14. 2. *d* seeing that the rectangle of D and F is given.) But D
e 1. *prop.* is given, therefore *e* the ratio of D to E is given, and
f 1. 6. as A is to C, so D is to F. But as A is to C, *f* so the square of A is to the rectangle of A and C, and also as D is to F, so the square of D is to the rectangle of D and F. Therefore as the square of A is to the rectangle of A and C, so the square of D is to the rectangle of D and F. But the rectangle of A and C is equal to the square of B, (seeing that A, B, and C, are proportional) and that of D and F to the square of E, therefore as the square of A is to the square of B, so the square of D is
g 12. 6. to the square of E: Wherefore *g* as A is to B, so D is
h 2. *def.* to E. But the ratio of D to E is given, therefore *h* also the ratio of A to B is given.

PROP. XXV.



If two lines AB and CD, given by position do intersect, the point E in which they intersect one another, is given by position.

Demonstr. For if it change its place, the one or the other of the lines AB and CD, would change its position: But so it is that by Supposition

sition it changeth not: Therefore *a* the point E is given by position. a 4. *def.*

PROP. XXVI.

If the extremities A and B, of a right line AB, are given in position, that same right line AB is given in position and in magnitude.

A———B

Demonstr. For if the point A remaining in its place, the position, or the Magnitude of the right line AB shall change, the point B will fall elsewhere. But so it is, that by Supposition it doth not fall elsewhere. Therefore the right line AB is given in position, and in magnitude.

PROP. XXVII.

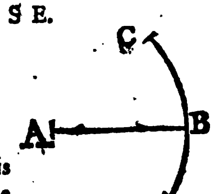
If one of the extremes A of a right line AB, given in position and magnitude, be given, the other extremity B shall be also given.

A———B

Demonstr. For if, the point A remaining in its place, the point B shall change and fall in some other place, either the position of the right line AB, or its magnitude would change: But so it is that according to the Supposition, neither the one nor the other doth change. Therefore the point B is given.

OTHERWISE.

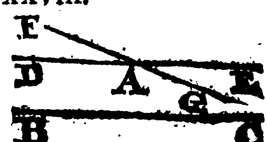
Constr. On the center A, with the distance AB, describe the circumference BC.



Demonstr. Therefore *a* that circumference BC is given by position. But the right line AB is also given by position; therefore the point *b* B is given. a 6. *def.* b 25. *prop.*

PROP. XXVIII.

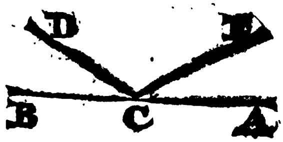
If through the given point A, there be drawn a right line DAE, against another right line BC, given in position, the right line DAE so drawn, is given in position.



Demonstr. For if it be not given, the point A remaining in its place, the position of the right line DAE may change: Let it then change if it be possible, and fall elsewhere, remaining parallel to BC, and let it be the line FAG: Therefore BC is parallel to the said line FAG,

a 13. def. b 30. 1. FAG. But *a* the same BC is also parallel to DAE: Therefore *b* DAE is parallel to the said line FAG, which is absurd; seeing they join together, and meet in A: Therefore the position of the right line DAE falls not elsewhere. Wherefore the said line DAE is given in position.

PROP. XXIX.



If to a right line AB, given in position, and to a point C given therein, there be drawn a right line CD, which shall make a given angle ACD, the line drawn CD is given in position.

Demonstr. For if it be not given in position, the point C remaining in its place, the position of the line CD observing the Magnitude of the angle ACD, will fall elsewhere. Let it fall elsewhere then if it be possible, and let it be CE. Therefore the angle ACD is equal to the angle ACE, the greater to the lesser, which is absurd. Therefore the position of the right line CD, shall not fall elsewhere; and therefore the said line CD is given in position.

PROP. XXX.

If from a given point A, be drawn to a right line BC, given in position, a right line AD, making a given angle ADB, that line drawn AD is given in position.

Demonstr. For if it be not given, the point A remaining in its place, the position of the right line AD changing, the Magnitude of the angle ADB, will change. Let it change then, and let it be the right line AE: Therefore the angle ADB is equal to the angle AEB, the greater *a* to the lesser, which is absurd. Therefore the position of the right line AD doth not change; and therefore the said line AD is given in position.

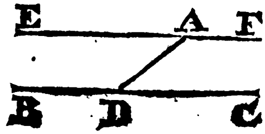
a 16. 1.

OTHERWISE.

Constr. Through the point A let there be drawn the line EAF, parallel to the right line BC.

Demonstr. Then seeing that through the given point A, and against the right line BC, given in position, there is drawn

drawn the right line EF, those lines EF and BC are parallels. But on the same lines doth also fall the right line AD. Therefore *b* the angle FAD is equal to the given angle ADB; and therefore it is also given. Wherefore to the right line EF given in position, and to the given point A therein, there is drawn the right line AD, making the given angle FAD. Therefore *c* the said line AD is given in position.



b 29. 1.

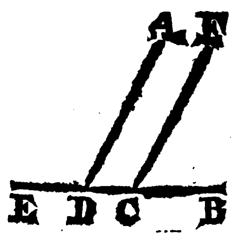
c 29. *prop.*

OTHERWISE.

Constr. In the line BCE, let there be taken the given point C, and through the same let there be drawn the line CF, parallel to the said DA.

Demonstr. Forasmuch as AD and CF are parallels, and that on them there doth fall the right line BCE, the angle FCB is equal *d* to the given angle ADB; and therefore it is also given. And seeing that the right line BC is given in position, and that to a given point C therein, there is drawn the right line FC, making the given angle FCB, that same line FC *e* is given in position. But through the given point A, opposite to the line FC given in position, there is drawn the line AD. Therefore the said line *f* AD is given in position.

d 29. 1.



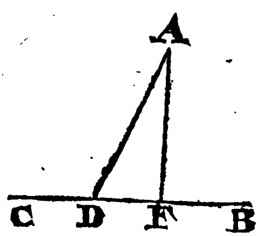
e 29. *prop.*

f 28. *prop.*

OTHERWISE.

Constr. In the right line BC assume some point at F, and draw AF.

Demonstr. Forasmuch as each point A and F is given, the right line AF is given *g* in position. But the line BC is also given in position. Therefore * the angle AFD is given. But by supposition, the angle ADF is given: Therefore DAF (which is the residue *h* of two right angles) is given; and seeing



g 26. *prop.*

h 32. 1.

I that

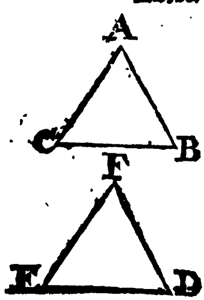
that to the right line AF given in position, and to the given point therein A there is drawn the right line DA,
i 29. prop. making the given angle DAF, *i* that same line DA is given in position.

Scholium.

* *EUCLIDE supposeth here that two right lines being given in position, and inclining to one another, do make a given angle, which some do demonstrate after this manner.*

Demonstr. Forasmuch as the two right lines given in position, do incline to one another, the inclination of those lines is given. But the angle is the inclination of the lines: Therefore the angle which makes the right lines given in position, and inclining to one another, is given.

Another thus demonstrateth it.



Constr. Let there be two right lines inclining to one another, as AB and CB, given in position, and in the line AB let there be taken a given point A, and in BC also some point, as C; and let the right line AC be drawn.

Demonstr. Seeing that as well the point B, as each of the points A and C is given, *k* the three
k 26. prop. right lines AB, BC, and AC, are given in Magnitude. Wherefore of three direct lines equal unto them, a triangle may be constituted: Let there then be made the triangle FDE, having the side FD equal to the side AB, the side FE equal to the side AC, and the base DE equal to the base BC.

Seeing then the angles comprised of equal right lines are equal, we have found the angle FDE equal to the
l 1. def. angle ABC; and therefore the same *l* angle ABC is given.

PROP.

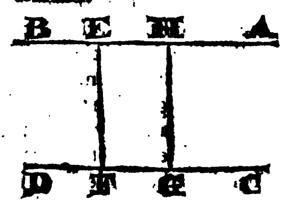
PROP. XXXI.

If from a given point A there be drawn to a right line given in position BC, a right line AD, given in magnitude, that line AD shall be also given in position.

Constr. From the center A, with the distance AD, let the circle DEF be described.

Demonstr. Forasmuch as the center A is given in position, and the semidiameter AD in magnitude, the circle DEF *a* is given in position. But the right line BC is also given in position: Therefore the point of intersection D *b* is given, and seeing that the point A is also given: *c* the right line AD is given in position. a 6. *def.* b 25. *prop.* c. 26. *prop.*

PROP. XXXII.



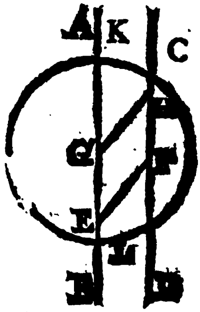
If unto parallel right lines AB and CD, given in position, there be drawn a right line EF, making the given angles BEF and EFD, the line drawn EF shall be given in magnitude.

Constr. For let there be taken in the line CD a given point G, and from that point let be drawn GH parallel to FE.

Demonstr. Forasmuch as the lines EF and HG are parallels, and that on them doth fall the line CD; *a* the angle EFD is equal to the angle FGH. But the angle EFD is given, therefore the angle FGH is also given. And forasmuch as to the right line CD given in position, and to the point G given in the same, there is drawn the right line GH, making the given angle FGH, *b* the said line GH is given in position. But AB is also given in position, therefore *c* the point H is given. But the point G is also given: Therefore *d* the line GH is given in Magnitude, and is *e* equal to EF. Wherefore *f* the said line is given in Magnitude. a 29. 1. b 29. *prop.* c 25. *prop.* d 26. *prop.* e 34. 1. f 1. *def.*

PROP.

PROP. XXXIII.



If unto parallel right lines AB and CD, given in position, there be drawn a right line EF given in magnitude, that line EF shall make the given angles BEF and DFE.

Constr. For let there be taken in the right line AB the point G, and through that point let there be drawn the line GH parallel to EF.

a 34. 1. *Demonstr.* Therefore EF is equal to the said *a* GH. But EF is given in Magnitude, therefore GH is also given in Magnitude. But the point G is given, and therefore, if on that point, with the distance GH, there

b 6. *def.* be described a circle, *b* that circle shall be given in position: Let it be then described, and let it be HKL, the said circle HKL is therefore given in position. But the line CD which doth cut the circumference KHL in H, is also given in position. Therefore the said point of inter-

c 25. *prop.* section H *c* is given. But the point G is given: There-

d 26. *prop.* fore *d* the right line GH is given in position. But the right

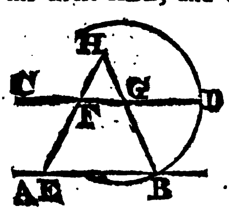
e *sch.* 30. *prop.* line CD is also given in position: Therefore *e* the angle GHF is given. But to that angle *f* the angle EFD is equal:

f 29. 1. Therefore the angle EFD is given; and therefore also the angle BEF; for that it is the residue of the sum of two

g 29. 1. *g* right angles.

OTHERWISE.

Constr. Let there be taken in the right line CD, the point G, and let GD be put equal to EF, then from the center G, with the distance GD, let there be described the circle HDB, and draw GB.



Demonstr. Forasmuch as the center G is given in position, and the semidiameter GD in magnitude, the circle BDH

h 6. *def.* *h* is given in position. But the line AB is also given in position:

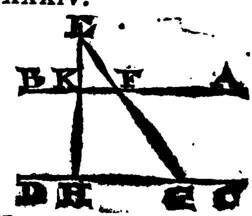
i 25. *prop.* Therefore *i* the point B is given. But the point G is also given,

k 26. *prop.* therefore *k* the right line GB is given in position. But the right line CD is also given

in

in position: Therefore *l* the angle BGD is given. l *sch.* 30.
Wherefore if EF be parallel to BG, the angle EFD *m* *prop.*
shall be given, and consequently also the other angle m 29. 1.
BEF. But the right lines BG and EF being not parallels, let them meet in the point H. Forasmuch as EB is parallel to FG, and EF is equal to GD, that is to say, to BG; also FH *n* shall be equal to GH (for EH and BH n 14. 5.
being cut proportionally *o* by the parallel FG, as EF is to o 2. 6.
FH, so is BG to GH, and by permutation, as EF is to BG, so is FH to GH:) Therefore *p* the angle HFG is p 5. 1.
equal to the angle HGF, but the said angle HGF is given (for that it is equal *q* to the given angle BGD:) There- q 15. 1.
fore the angle HFG is also given. But to that angle the angle BEF is equal; and therefore is given, as also the remaining angle EFG.

PROP. XXXIV.

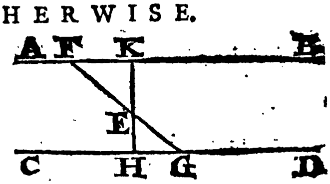


If from a given point E, there be drawn unto parallel right lines AB and CD, given in position, a right line EFG, that right line EFG shall be divided in a given ratio (to wit) as EF to FG.

Constr. For from the point E let there be drawn the line EH, perpendicular to the line CD.

Demonstr. Forasmuch as from the given point E there is drawn to the line CD the right line EH, making the given angle EHG, *a* the said line EH is given in position; a 30. *prop.*
but both the one and the other lines AB and CD is also given in position. Therefore *b* the points of intersection b 25. *prop.*
K and H, are given. But the point E is also given: Therefore *c* each line EK and KH is given. Wherefore c 26. *prop.*
d the ratio of the said EK to KH is given. But as d 1. *prop.*
EK is to KH, so is EF to FG; (for in the triangle GEH the line KF being parallel to HG, the sides EH and EG are cut proportionally:) Therefore the ratio of the said EF to FG is given.

OTHERWISE.



Constr. To the parallel right lines given in position, AB and CD, let there be drawn from the point E the right line FEG: I say, that the ratio of GE to EF is given.

Demonst. For from the point E let there be drawn to CD the perpendicular EH, and produced to the point K; seeing therefore that from the point E to the right line CD, given in position, there is drawn the line EH, making the given angle EHG, *a* the said line EH is given in position. But each line AB and CD is also given in position: Therefore *b* each point of intersection H and K is given. But the point E is also given, therefore *c* each of the lines EH and IK is given in Magnitude; and therefore *d* the ratio of the said EH to EK is given. But *e* as EH is to FK so is EG to EF (for the opposite angles at the point E being equal, and the lines AB and CD parallels, the triangles EHG and EKF are equiangled; and therefore as EH is to EG, so is EK to EF; and by permutation as EH to EK, so is FG to EF.) Therefore the ratio of the said lines EG to EF is given.

a 30. *prop.* b 25. *prop.* c 26. *prop.* d 1. *prop.* e 4. 6.

PROP. XXXV.

If from a given point A, to a right line BC, given in position, there be drawn a right line AD, which let be divided in E, in a given ratio (to wit) as AE to ED, and that by the point of section E there be drawn a right line FEG, opposite to the right line BC, given in position, the line FG drawn shall be given in position.

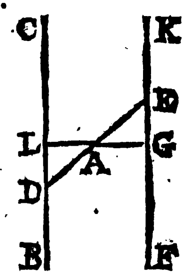
Constr. For from the point A, let there be drawn the line AH, perpendicular to the line BC.

Demonstr. For seeing that from the given point A there is drawn to BC given in position, the right line AH making the given angle AHD, *a* the said line AH is given in position. But BC is also given in position: Therefore *b* the point H is given. But the point A is also given: Therefore *c* the line AH is given in magnitude and in position. And seeing that *d* as AE is to ED, so is AK to KH, and that the ratio of AE to ED is given, also the ratio of AK to KH is given; and by compounding, *e* the ratio of AH to AK is given. But AH is given in Magnitude: Therefore *f* also AK is given in Magnitude. But AK is also given in position, and the point A is given: Therefore *g* the point K is also given, and seeing that by the said

a 30. *prop.* b 25. *prop.* c 26. *prop.* d 2. 6. e 6. *prop.* f 2. *prop.* g 27. *prop*

given point K there is drawn the line FG, opposite to the right line BC given in position; the said line FG h* is given in position. h 28. prop.

PROP. XXXVI.



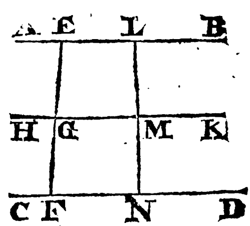
If from a given point A, there be drawn to a right line BC given in position, a right line AD, and to it be added a right line AE, having to the same AD a given ratio, and that through the extremity E of the added line AE, there be drawn a right line FEK, opposite to the line BC, given in position, that same line FEK shall be given in position.

Constr. For from the point A let there be drawn to the line BC, the perpendicular AL, and let it be prolonged to the point G.

Demonstr. Forasmuch as from the given point A, there is drawn to the right line BC, given in position, the righ line GL, which makes the given angle GLD, *a* that line GL is given in position. But BC is also given in position, therefore *b* the point L is given; and seeing that the point A is also given, the line *c* AL is given. But forasmuch as the ratio of AE to AD, is given; and that *d* as the said AE is to AD, so is AG to AL; (because the triangles ALD and AGE are equiangled) the ratio of AG to AL is also given. But AL is given in Magnitude: Therefore *e* AG is given in Magnitude. But it is also given in position, and the point A is given: Therefore *f* the point G is also given. And seeing that by the same given point G there is drawn the line FK, opposite to the right line BC, given in position, *g* the said line FK is given in position.

a 30. prop.
b 26. prop.
c 26. prop.
d 4. 6.
e 2. prop.
f 27. prop.
g 28. prop.

PROP. XXXVII.



If unto parallel right lines AB and CD, given in position, there be drawn a right line EF, divided in the point G, in a given ratio, (to wit, of EG to GF;) and if through the point of section G, there be drawn opposite to the right lines AB or CD, given in position, a right line HGK, that line drawn shall be given in position.

 Constr.

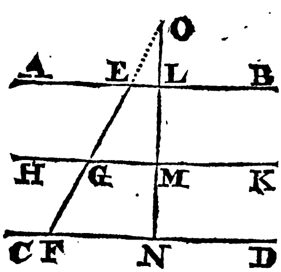
Constr. For let there be taken in the line AB the given point L, and from that point let there be drawn the line LN, perpendicular to CD.

Demonstr. Seeing that from the given point L, there is drawn to the right line CD, the line LN, making the
a 30. *prop.* given angle LND, the said LN *a* is given in position. But CD is also given in position: Therefore the point
b 25. *prop.* N *b* is given. But the point L is also given: Therefore
c 26. *prop.* *c* the line LN is given; and seeing that the ratio of FG to GE is given, and that * as FG is to GE, so is NM to ML, the ratio of the said NM to ML is given; and
d 6. *prop.* by compounding, *d* the ratio of LN to LM is also given. But LN is given in Magnitude, therefore ML
e 2. *prop.* is *e* given in Magnitude. But it is also given in position, and the point L is given: Therefore the point M
f 27. *prop.* *f* is also given. And considering that through the said point M there is drawn the right line KH, opposite to the right line CD, given in position, the said line KH is also given in position.

Scholium.

* *EUCLIDE supposeth here, that as FG is to GE, so NM is to ML; but by another it is thus demonstrated.*

The lines EF and LN are parallels or not parallels: Let them in the first place be parallels, and forasmuch as by Construction the lines EL, FN, EF, and LN, are parallels, EN shall be a parallelogram; and therefore the side EF is equal to the side LN. Again, seeing that MG is parallel to NF, and GF to MN, GN shall be also a parallelogram; and therefore the side GF is equal to the side MN. Wherefore the equal sides EF and LN, shall have to the equal sides FG
g 7. 5. *and MN,* g *one and the same ratio. Therefore as EF is*
h 17. 5. *to FG, so is LN to MN; and by dividing,* h *as GE to GF, so is LM to MN.*



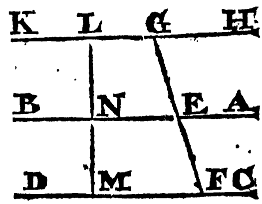
Now suppose that the lines EF and LN are not parallels, but that they meet in the point O. Forasmuch as in the triangle OFN there is drawn
i 2. 6. *HK, parallel to FN one of the sides;* i *the sides OF and ON are divided proportionally; and therefore as FG is to GO, so is NM to MO. Again, seeing that in the triangle OGM there is drawn*

drawn EL, parallel to the side GM, the sides OG and OM are divided proportionally: Wherefore k *as OE is to EG, so is OL to LM, and by compounding,* l *as OG is to EG, so is OM to LM; but it hath been demonstrated that as FG is to GO, so is NM to MO; therefore in ratio of equality,* m *as FG is to GE, so is NM to ML.*

k 2. 6.
l 18. 5.
m 22. 5.

PROP. XXXVIII.

If unto parallel right lines AB and CD, there be drawn a right line EF, and that to it there be added some other right line EG, which hath a given ratio to the same EF; and if through the extremity G of the added line EG, there be drawn a right line HK, against the parallels given in position AB and CD, the line drawn HK shall be also given in position.



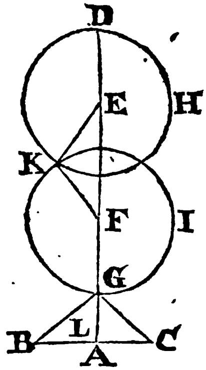
Constr. For let there be taken in the line AB, the given point N, and from thence let there be drawn to CD the perpendicular NM, and let it be prolonged to the point L

Demonstr. Forasmuch as from the given point N there is drawn to the right line CD, given in position, the right line NM making a given angle NMF, the said angle NMF *a* is given in position. But the line CD is also given in position: Therefore *b* the point M is given. But the point N is also given: Therefore *c* the line NM is given, and for that the ratio of EG to EF is given; and that *d* as EG to EF, so is LN to NM, the ratio of LN to NM is also given: But NM is given, therefore LN is *e* also given. But the point N is given: Therefore *f* the point L is also given. Seeing then that by the given point L there is drawn the right line HK, opposite to the line AB given in position, *g* the said line AK is also given in position.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d *sch.* 37. *prop.*
e 2. *prop.*
f. 27. *prop.*
g 28. *prop.*

PROP. XXXIX.

If all the sides of a triangle ABC are given in magnitude, the triangle is given in kind.



Constr. For, let there be exposed the right line DG given in position, ending in the point D; but being infinite towards the other part G, and therein let be taken DE, equal to AB.

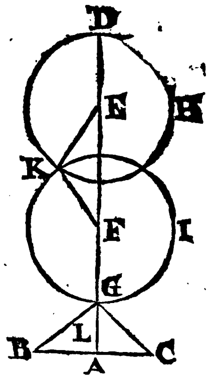
Demonstr. Now seeing the said AB is given in magnitude, DE is so also; but the same DE is also given in position, and the point D is given: Therefore
a 27. prop. *a* the point E is given.

Again, Let EF be put equal to BC; and seeing that BC is given in magnitude, EF shall be so also. But the said EF is in like manner given in position, and the point E
b 27. prop. is given: Therefore *b* the point F is given.

Furthermore, Let FG be taken equal to AC. Now forasmuch as the said AC is given in magnitude, FG is so also. But FG is also given in position, and the point F is given: Therefore the point G is also given. Now from the center E, with the distance ED, let there be de-
c 6. def. scribed the circle DHK, *c* and that circle shall be given in position. Again, on the center F, and distance FG,
d 6. def. let there be described the circle GLK. Therefore *d* the
e 25. prop. said circle GLK is given in position; and therefore *e* the point of Intersection K is given. But each of the points
f 26. prop. E and F is given: Therefore each line *f* EK, EF, and FK, is given in position and magnitude. Therefore the triangle FK is given * in kind; but it is equal and alike to the triangle ABC; and therefore the triangle ABC is also given in kind.

Scholium.

Scholium.



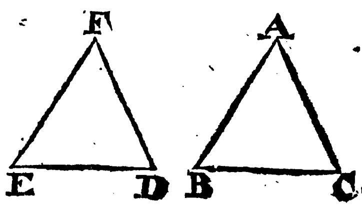
* *EUCLIDE supposeth here, that a triangle, whose sides are given in magnitude and position, is given in kind; but the antient Interpreters demonstrate it in a manner thus. Forasmuch as the right lines* KE *and* EF *are given, g the ratio which they have to one another is given. Also the right lines* EF *and* FK *being given, their ratio is also given; and in like manner, the ratio of the said* EK *and* FK *is given. Again, seeing that the same lines* KE *and* EF *are given in position, h the angle* KEF *is given in magnitude: Moreover, the right lines* EF *and* FK *being given in position, the angle* EFK *is given in magnitude, as is also the residue* EKF, *and so in the triangle* EKF *are all the angles given, and also the ratio's of the sides: Therefore i the said triangle* EKF *is given in kind.*

g 1. *prop.*

h *sch.* 30. *prop.*

i 5. *def.*

PROP. XL.



If the angles of a triangle ABC, *are given in magnitude, the triangle is given in kind.*

Constr. Let there be exposed the right line DE, given in position and in magnitude; and let there be constituted at the point D the angle EDF, equal to the angle CBA; and at the point E the angle DEF, equal to the angle BCA; therefore the third angle BAC is equal to the third angle DFE.

Demonst. For each of the angles constituted in the points A, B, and C, is given: Therefore each of those which are posited in the points D, F, and E, is also given; and seeing that to the right line DE given in position, and to the point D given therein, there is drawn the right line DF, which makes the given angle EDF, a the

a 29. *prop.*

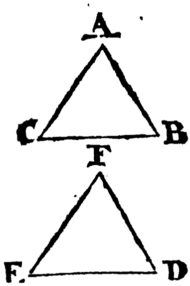
the line DF is given in position; and for the same reason,
b 25. prop. the line EF is given in position: Therefore *b* the point F is given in position. But each of the points D and
c 26. prop. E is given: Therefore *c* each of the lines DF, DE, and EF, is given in magnitude. Wherefore the triangle DFE is given in kind; and is alike to the triangle ABC: Therefore the triangle ABC is given in kind.

PROP. XLI.

If a triangle ABC, hath one angle BAC given, and that the two sides BA and AC, which do constitute it, have to one another a given ratio, the triangle is given in kind.

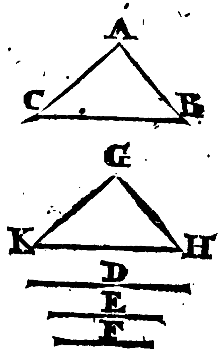
Constr. For, let there be exposed the right line DF given in magnitude and position. And thereon, and at the given point F, let there be constituted the angle DFE equal to the angle BAC.

Demonstr. Now the angle BAC is given: Therefore also the angle DFE is given, and seeing that to the right line DF given in position, and from the given point F therein is drawn a right line FE, making the
a 29. prop. given angle DFE, *a* the said line FE is given in position. But seeing that the ratio of AB to AC is given, let the same be made of DF to FE, then let DE be drawn. Therefore the ratio of DF to FE is given. But DF is given: Therefore *b* FE is
b 2. prop. given in magnitude. But the same FE is also given in position, and the point F is given. Therefore *c* the point E is also given.
c 27. prop. But each of the points D and F is given: Therefore *d* each of the
d 26. prop. right lines DF, FE, and DE is given in position and magnitude. Wherefore *e* the triangle DEF is given in
e 39. prop. kind. And seeing that the two triangles ABC and DEF have an angle equal to an angle, that is to say, the angle BAC to the angle DFE, and the sides which con-
f 6. 6. stitute those equal angles, proportional; *f* the triangle ABC is alike to the triangle DEF. But the triangle DEF is given in kind: Therefore the triangle ABC is given in kind.



PROP.

PROP. XLII.



If the sides of a triangle ABC, are to one another in given ratio's, the triangle ABC is given in kind.

Constr. For, let there be exposed the right line D, given in magnitude, and seeing that the ratio of BC to AC is given, let the same be made of D to E.

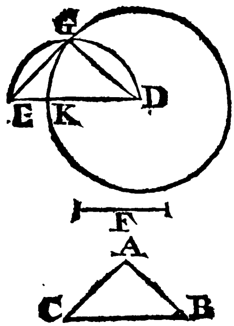
Demonstr. Now D is given, therefore *a* E is also given. Again, seeing that the ratio of AC to AB is given, let the same be made of E to F. Now E is given, therefore *b* F is also given. Now of three right lines, equal to the three given right lines D, E, and F, (and of which three lines, two of them, in what manner soever they be taken, are greater than the other,) let there be constituted the triangle GHK, in such sort as D may be equal to HK; but E is equal to KG, and GH equal to F; therefore each of the said lines HK, KG, and GH, is given in magnitude: Wherefore *c* the triangle HGK is given in kind. And seeing that as BC is to CA, so is D to E, and that D is equal to HK, and E to KG, as BC is to CA, so HK is to KG. Again, seeing that as CA is to AB, so is E to F, and that E is equal to KG, and F to GH; as CA is to AB, so is KG to GH. But it hath been demonstrated, that as BC is to CA, so is HK to KG: Therefore in ratio of equality, as BC is to AB, so is HK to GH. Therefore *d* the triangle ABC is also given in kind.

a 2. *prop.*
b 2. *prop.*
c 39. *prop.*
d 5. 6.

PROP.

PROP. XLIII.

If the sides BC *and* BA, *about one of the acute angles of a rectangled triangle* ABC, *have to one another a given ratio, that triangle is given in kind.*

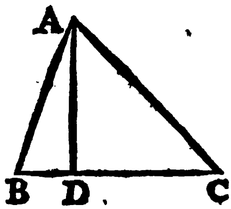


Constr. Let there be exposed the right line DE given in magnitude and position, and on it let there be described the semicircle DGE: There-
a 6. def. fore *a* the semicircle DGE is given in position.

Demonstr. For the line DE being given, and divided in two equal parts, the center of the said circle is given in position, and the semidiameter in magnitude. And forasmuch as the ratio of BC to BA is given, let the same be made of DE to F; Therefore the ratio of DE to F is given. But DE is
b 2. prop. given, therefore F *b* is also given. Now BC is greater
c 19. 1. than *c* AB: Therefore ED is *d* also greater than F.
d 14. 5. Let DG be fitted equal to F, and let EG be drawn; then on the center D, with the distance DG, let the
e 6. def. circle GK be described. Now that circle *e* is given in position, seeing that the center D is given, and the semidiameter DG is also given in magnitude. But the semi-
f 25. prop. circle DGE is also given in position: Therefore *f* the point of intersection G is given. But the points D and
g 26. prop. E are also given, therefore *g* each of the right lines DE, DG, and EG, is given in position and magnitude.
h 39. prop. Wherefore *h* the triangle DGE is given in kind. And seeing that the triangles ABC and DGE have an angle equal to an angle, to wit, the right angle BAC to the
i 31. 3. right angle *i* DGE, and the sides about the angles CBA and EDG proportional. But each of the others ACB and DEG are less than a right angle: Those triangles
k 7. 6. ABC and DEG *k* are alike. But the triangle DGE is given in kind: Therefore the triangle ABC is also given in kind.

PROP. XLIV.

If a triangle ABC, hath one angle B given, and that the sides BA and AC, about another angle BAC, have to one another a given ratio, the triangle ABC is given in kind.



Constr. Now the given angle B is either acute or obtuse, (for it was a right angle in the foregoing proposition.) Let it be in the first place acute, and from the point A let AD be drawn perpendicular to BC.

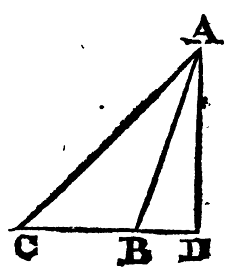
Demonstr. Therefore the angle ADB is given: But the angle B is also given; and therefore the third angle BAD is given: Wherefore *a* the triangle ABD is given in kind; and therefore *b* the ratio of BA to AD is given. But the ratio of the same BA to AC is also given: Therefore *c* the ratio of AD to AC is given, and the angle ADC is a right angle: Wherefore the triangle *d* ACD is given in kind: Therefore *e* the angle C is given. But the angle B is also given; and therefore the other angle BAC is given: Therefore *f* the triangle ABC is given in kind.

a 40. *prop.*
b 3. *def.*
c 8. *prop.*
d 43. *prop.*
e 3. *def.*
f 40 *prop.*

Constr. Now let the angle ABC be obtuse, and on the side CB prolonged, let there be drawn the perpendicular AD.

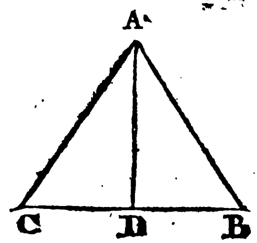
Demonstr. Forasmuch as the angle ABC is given, the angle ABD, which follows it, shall be given. But the angle ADB is also given: Therefore the third angle DAB is given. Wherefore *g* the triangle ABD is given in kind; and therefore *h* the ratio of DA to AB is given. But the ratio of AB to AC is also given: Therefore *i* the ratio of DA to AC is given, and the angle D is a right angle. Therefore the triangle DAC is given in kind, and therefore the angle ACB is given. But the angle ABC is also given: Therefore the third angle BAC is given. Wherefore the triangle ABC is given in kind.

g 40. *prop.*
h 3. *def.*
i 8. *prop.*



PROP.

PROP. XLV.



If a triangle ABC hath one angle BAC given, and that the line compounded of the two sides AB and AC, about the said given angle BAC, hath to the other side BC a given ratio, the triangle ABC is given in kind.

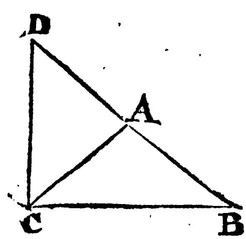
Constr. For, let the angle BAC be divided into two
a 7. *prop.* equal parts by the line AD, therefore *a* the angle CAD is given.

b 3. 6. *Demonstr.* Seeing that as AB is to AC, so *b* is BD to
c 18. 5. CD; by compounding, *c* as the line compounded of CAB is to CA, so is BC to CD, and by permutation, as the line compounded of CAB is to CB, so is CA to CD. But the ratio of the line compounded of CAB to BC, is given; therefore the ratio of CA to CD is also
d 44. *prop.* given, and the angle CAD is given. Therefore *d* the triangle ACD is given in kind, and therefore the angle C is given. But the angle BAC is also given: There-
e 40. *prop.* fore the third angle B is given: Wherefore *e* the triangle ABC is given in kind.

OTHERWISE.

Constr. Let BA be prolonged directly unto the point D, in such sort as that AD may be equal to AC, and let CD be joined.

Demonstr. Forasmuch as the ratio of the line compounded of CAB to CB is given, and that AD is equal to AC, the ratio of the whole line BD to BC is given. But the angle ADC is also given, for it is the half of the given angle BAC (for that the said
f 32. 1. angle BAC *f* is equal to the two internal angles ACD and
g 5. 1. ADC, which are *g* equal to one another, being the sides AC and AD are equal:) Wherefore the triangle BDC

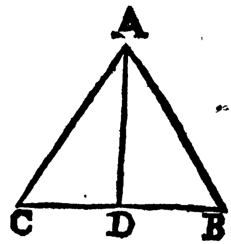


h 44. *prop.* *h* is given in kind, and therefore the angle B is given. But the angle BAC is also given. Therefore the re-
i 40. *prop.* maining angle ACB is given: Wherefore *i* the triangle ABC is given in kind.

PROP.

PROP. XLVI.

If a triangle ABC hath one angle B given, and that the line CAB compounded of the two sides AC and AB about another angle BAC hath to the other side BC a given ratio, the triangle ABC is given in kind.



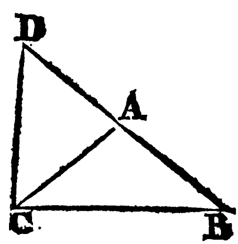
Constr. For let the angle BAC be divided into two equal parts, by the line AD.

Demonstr. Therefore (as hath been shewn in the foregoing Proposition) the compound line CAB is to CB, as AB is to BD. But the ratio of the said compound line CAB to CB is given: Therefore also the ratio of AB to BD is given. But the angle B is also given: Therefore the triangle ABD *a* is given in kind; and therefore *b* the angle BAD is given. But the angle BAC is double to that of BAD; and therefore it is also given. Therefore the third angle C is given. Wherefore the triangle ABC is given in kind.

a 41. *prop.*
b 2. *def.*

OTHERWISE

Constr. Let BA be prolonged directly, and let AD be put equal to AC, and let CD be joined.

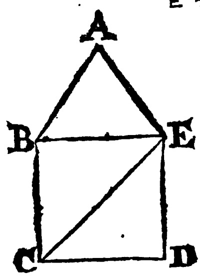


Demonstr. Forasmuch as the ratio of the line compounded of CAB to CB is given, and that AD is equal to AC, the ratio of BD to BC is given; and the angle B is also given: Therefore the triangle CDB *c* is given in kind; and therefore *d* the angle D is given: Therefore the angle BAC which is double to BDC, is also given: Wherefore the other angle ACB is given; and therefore the triangle ABC is given in kind.

c 41. *prop.*
d 3. *def.*

PROP.

PROP. XLVII.



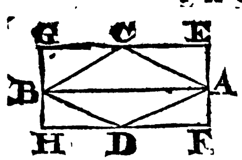
Rectiline figures, as AB CDE, given in kind, are divided into triangles given in kind.

Constr. For let the right lines EB and EC be drawn.

Demonstr. Forasmuch as the rectiline figure ABCDE is given in kind, the angle *a* BAE is given, and the ratio of the side AB to AE is also given: Therefore *b* the triangle BAE is given in kind. Wherefore the angle ABE is given. But the whole angle ABC is also given: Therefore *c* the remaining angle EBC is given. But the ratio of the side AB to the side BE, and also that of AB to BC is given: Therefore *d* the ratio of BC to BE is given, and the angle CBE is also given: Therefore *e* the triangle BCE is given in kind. By the same way it may be demonstrated that the triangle CDE is given in kind. Therefore rectiline figures given in kind divide themselves into triangles given in kind.

a 3. *def.*
b 41. *prop.*
c 4. *prop.*
d 8. *prop.*
e 41. *prop.*

PROP. XLVIII.



If on one and the same right line AB, are described triangles, as ACB and ABD, given in kind, those triangles shall have to one another a given ratio, as ACB to ABD.

Constr. For from the points A and B, let there be drawn at right angles on the line AB, the lines AE and BG, and prolonged unto the points F and H; through the points C and D, let there be drawn the lines ECG and FDH, parallel to AB.

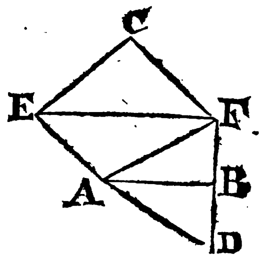
Demonstr. Forasmuch then as the triangle ABC is given in kind, *a* the ratio of CA to BA is given, and the angle CAB is also given; but the angle BAE is given: Therefore the remaining angle CAE is also given; but the angle CAE is given; and therefore the other angle ACE is also given. Wherefore *b* the triangle AEC is given in kind. Now the ratio of EA to AB *c* is given; (for *d* the ratio of EA to AC, and that of AC to AB is given;) and in like manner, the

a 3. *def.*
b 40. *prop.*
c 8. *prop.*
d 3. *def.*

the ratio of FA to AB is given. Therefore *e* the ratio of EA to AF is given; but as AE is to AF, so *f* the parallelogram AH to the parallelogram AG; but ACB is *g* the half of AH, and ADB the half of AG; therefore the ratio of the triangle ACB to the triangle ADB is given; for it is the same ratio with that of AH to AG *h*; that is to say, of EA to AF, which is given. e 8. *prop.* f 1. 6. g 41. 1. h 15. 5.

PROP. XLIX.

If on one and the same right line AB there are described any two rectiline figures AECFB and ADB, given in kind, they shall have to one another a given ratio (to wit) AECFB to ADB.

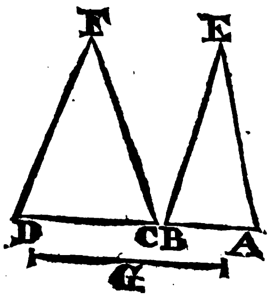


Constr. For let the lines FA and FE be drawn: Therefore each of the triangles *a* ABF, AFE, and ECF is given in kind. a 47. *prop.*

Demonstr. Seeing that on one and the same right line EF there are described the triangles ECF and EAF, given in kind; the ratio of ECF to EAF *b* is given. Therefore by compounding, *c* the ratio of AECF to EAF is given. But the ratio of the said EAF to FAB is given, *d* because they are triangles given in kind, described on one and the same right line AF: Therefore *e* the ratio of AECF to FAB is given. Wherefore by compounding, *f* the ratio of AECFB to FAB is given. But the ratio of the same FAB to ABD *g* is given: Therefore *h* the ratio of AECFB to ABD is also given. b 48. *prop.* c 6. *prop.* d 48. *prop.* e 8. *prop.* f 6. *prop.* g 48. *prop.* h 8. *prop.*

PROP.

PROP. L.

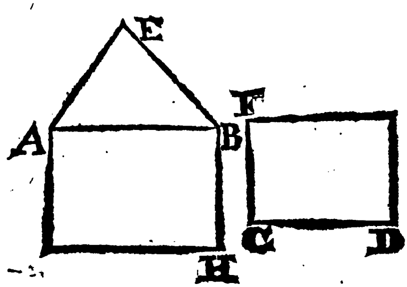


If two right lines AB and CD, have to one another a given ratio, and that on those lines there be described rectiline figures AEB and CFD, alike, and alike posited, they will have to one another a given ratio.

Demonstr. To the two lines AB and CD, let there be taken a third proportional G: Therefore as AB is to CD, so is CD to G. But the ratio of AB to CD is given: Therefore the ratio of CD to G is also given: Wherefore *a* the ratio of AB to G is given. But *b* as AB is to G, so is AEB to CFD: Therefore the ratio of the same AEB to CFD is given.

a 8. *prop.*
b *cor.* 19. 20. 6.

PROP. LI.



If two right lines AB and CD have to one another a given ratio, and that upon them there be described any rectiline figures AEB and CFD, given in kind, they will have to one another a given ratio, (to wit, that of AEB to CFD.)

Constr. For on AB let the rectangled figure AH be described alike, and alike posited to DF.

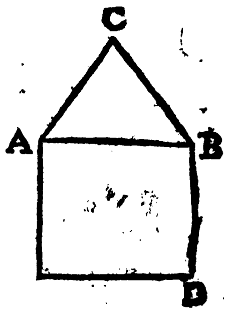
Demonstr. Now DF is given in kind: Therefore also AH is given in kind. But AEB is also given in kind, and described on the same line AB: Therefore *a* the ratio of AEB to AH is given: And seeing that the ratio of AB to CD is given, and that on those lines are described the rectiline figures AH and DF alike, and alike posited, the ratio *b* of the said line AH to DF is given. But the ratio of AEB to AH is also given: Therefore the ratio *c* of AEB to DF is given.

a 49. *prop.*
b 50. *prop.*
c 8. *prop.*

PROP.

PROP. LII.

If on a right line AB, given in magnitude, there be described a figure ACB, given in kind, that figure ACB is given in magnitude.



Constr. For on the same line AB, let the square AD be described. Therefore AD is given in kind * and in magnitude.

Demonstr. Seeing that on the right line AB, are described the two rectiline figures ACB and AD, given in kind, *a* the ratio of ACB to AD is given: Therefore *b* ACB is given in magnitude.

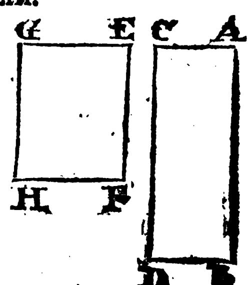
a 29. *prop.*
b 2. *prop.*

Scholium.

* *The antient interpreter hath noted here that every square is given in kind; for that all the angles thereof are given; being all equal and right angles: But also the ratio's of the sides are given; for those sides being all equal, their ratio's are also equal. Moreover, whensoever a square is exposed, a square equal thereto may be exhibited; and therefore the square is given in magnitude, as also each side thereof.*

PROP. LIII.

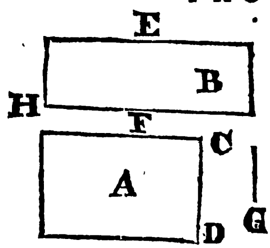
If there are two figures AD and EH, given in kind, and that one side BD of the one, hath to a side EH of the other, a given ratio; the other sides shall have also to the other sides given ratio's.



Demonstr. For seeing that the ratio of BD to EH is given, and also that *a* of BD to BA, *b* the ratio of the said AB to EH is given. But the ratio of the same EH to EF *c* is also given: Therefore *d* the ratio of AB to EF is given. In like manner also the ratio's of the other sides to the other sides are given.

a 3. *def.*
b 8. *prop.*
c 3. *def.*
d 8. *prop.*

PROP. LIV.



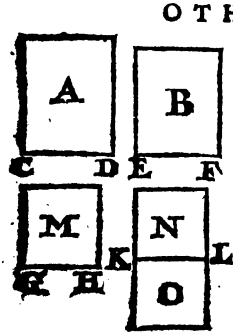
If two figures A and B given in kind, have to one another a given ratio, also their sides shall be to one another in a given ratio.

Constr. For either the figure A is alike and alike posited to B, or is not: Let it in the first place be alike, and alike posited; and let there be taken the line G, a third proportional to the lines CD and EF.

a cor. 19. 20. 6. *Demonstr.* As CD is to G, *a* so is A to B. But the ratio of A to B is given; therefore also the ratio of CD to G is given. And seeing that CD, EF, and G, are
b 24. prop. proportional, *b* also the ratio of CD to EF is given. But
c 53. prop. A and B are given in kind: Therefore *c* the other sides shall have given ratio's to the other sides.

Now let the figure A be not alike to the figure B, and let there be described on EF the figure EH, alike and alike posited to A: Therefore the figure EH is given in kind; but the figure B is also given in kind: There-
d 49. prop. fore *d* the ratio of B to EH is given; and therefore the
e 8. prop. ratio of A to the same EH *e* is also given: But A is alike to EH: Therefore (by what is abovesaid) the ratio of CD to EF is given; and in like manner the ratio of the other sides to the other sides is given.

OTHERWISE.



Constr. Let there be exposed the given line GH: Now either the figure A is alike to the figure B, or not. Let it in the first place be alike, and let it be as CD it to EF, so is GH to LK; then on GH and LK let the figures M and N be described alike, and alike posited to the said A and B, which figures M and N shall be consequently given in kind.

Demonstr. Therefore seeing that as CD is to EF, so is GH to LK, and that on

those

those lines CD, EF, GH, and LK, are described the figures A, B, M, and N, alike and alike posited; *f* as A is to B, so is M to N. But the ratio of A to B is given: Therefore the ratio of M to N is given. But *g* M is given, considering that it is a figure given in kind, described on a right line given in magnitude; therefore N is also given.

f 22. 6.
g 52. *prop.*

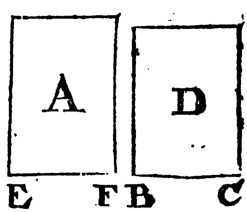
Constr. 2. Now, on LK let the square O be described: Therefore *h* the figure O is given in kind. h *sch.* 52. *prop.*

Demonstr. 2. Wherefore the ratio of O to N is given. But N is given: Therefore O is given; and consequently *i* also KL. But GH is given: Therefore *k* the ratio of GH to KL is given. But as GH is to LK, so is CD to EF. Therefore the ratio of CD to EF is given; and therefore the figures A and B being given in kind, *l* the other sides of the same figures shall also have to the other sides given ratio's. But if the figures be not alike, the latter part of the demonstration here above must be observed.

i *sch.* 52. *prop.*
k 3. *prop.*
l 53. *prop.*

PROP. LV.

If a space A be given in kind, and in magnitude, the sides thereof shall be given in magnitude.

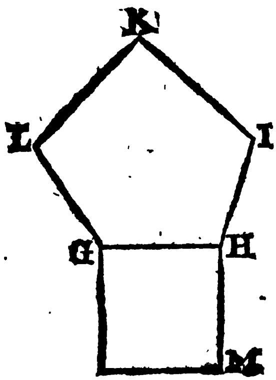


Constr. For, let the right line BC, given in position and in magnitude, be exposed; and thereon let there be described the space D, alike and alike posited to A; therefore the said space D is given in kind.

Demonstr. For that it is described on the line BC, given in magnitude, it is also *a* given in magnitude. But the figure A is also given: Therefore *b* the ratio of A to D is given. But those figures A and D are given in kind: Therefore *c* the ratio of the line EF to the line BC is given. But BC is given: Therefore *d* EF is also given. But the ratio of the same EF to FG is given: Therefore *e* FG is given. And by the same ways of reasoning it may be demonstrated that each of the other sides are given in magnitude.

a 52. *prop.*
b 3. *prop.*
c 54. *prop.*
d 3. *def.*
e 2. *prop.*

OTHERWISE.



Constr. Let the space GHIKL be given in kind and in magnitude: I say that the sides thereof are given in magnitude. For on the right line GH let there be described the square GM; therefore *f* GM is given in kind.

f sch. 52. prop.

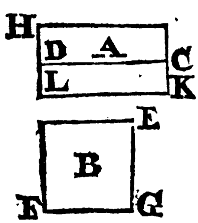
Demonstr. But the space GHIKL is also given in kind: Therefore *g* the ratio of the same space GK to GM is given. But GK is given in magnitude: Therefore *h* GM is also given in magnitude; and seeing that GM is the square of the line GH, *i* that line GH is given in magnitude. Wherefore in like manner, each of the other lines HI, IK, KL, and LG, is given.

g 49. prop.

h 2. prop.

i sch. 52. prop.

PROP. LVI.



If two equiangled parallelograms A and B, have to one another a given ratio, as one side CD of the first A, is to one side FG, of the second B; so the other side GE, of the second B, is to that to which DH the other side of the first A, hath the given ratio that the parallelogram A hath to the parallelogram B.

Constr. For let HD be prolonged directly to L, so that as CD is to FG, so HD may be to DL; and finish the parallelogram DK.

Demonstr. Seeing that as CD is to FG, so HD is to DL, and *a* that CD is equal to KL; as LK is to FG, so is GE to DL; and thus the sides about the equal angles DLK and EGF are reciprocally proportional: Wherefore *b* DK is equal to B; and therefore seeing the ratio of A to B is given, and that B is equal to DK, the ratio of A to DK it given. But as *c* A is to DK (that is to B) so is HD to DL: therefore the ratio of HD to DL is also given: and seeing that as CD is to FG, so GE is to DL, and that the right line

a 34. 1.

b 14. 6.

c 1. 6.

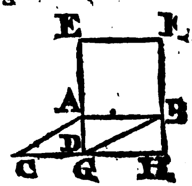
HD hath to DL a given ratio; to wit, that which the space A hath to the space B; as CD is to FG, so GE is to that to which HD hath the given ratio that the space A hath to the space B, that is to say, the ratio of HD to DL.

PROP. LVII.

If a given space AD be applied to a given right line AB in a given angle CAB, the breadth CA of the application is given.

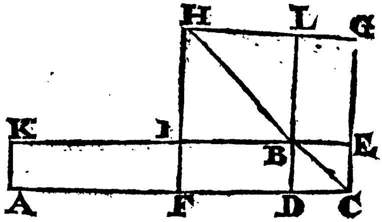
Constr. For on AB, let there be described the square AF; therefore *a* the same AF is given: Let the lines EA, FB, and CD, be prolonged to the points G and H. a *sch.* 52. *prop.*

Demonstr. Seeing therefore that each space AD and AF is given, their ratio is also given. But *b* AD is equal to AH: Therefore the ratio of AF to AH is given: Wherefore the ratio of EA to AG is given. (For *c* it is the same with that of AF to AH.) But EA is equal to AB; therefore the ratio of AB to AG is given. Now seeing that the angle CAB is given, and the angle GAB also given, the residue CAG is given. But the angle CGA is also given, being a right angle: Therefore the remaining angle ACG is given. Wherefore the triangle *d* CAG is given in kind: Therefore the ratio of CA to AG is given. But the ratio of AB to the same AG is also given: Therefore the ratio of CA to AB is given; and the said AB is given: Wherefore CA is also given. b 36. 1. c 1. 6. d 40. *prop.*



PROP. LVIII.

If a given space AB, be applied to a given right line AC, wanting by a figure DE, given in kind, the breadths of the defects are given.

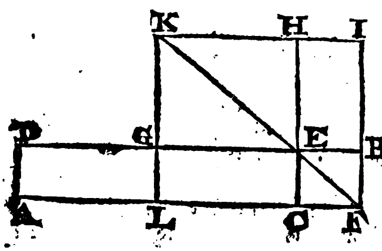


Constr. For let AC be divided in two equal parts in the point F: Therefore as well AF as FC is given. On the said line FC let there be described the rectangled figure FG alike and alike posited to DE. Therefore FG is given in kind.

Demonstr. Seeing the figure FG is described on the right line FC given in magnitude, the said rectiline FG is *a* also given in magnitude. But FG is equal to AB and IL; (for *b* AI and FE being equal, and *c* FB and BG also equal, the Gnomon ICL is equal to AB; and therefore their added figure IL common to both, FG shall be equal to AB and IL:) Therefore the figures AB and IL together are given in magnitude. But AB is given in magnitude: Therefore *d* the remaining figure IL is also given in magnitude. But it is also given in kind, seeing it is *e* alike to DE: Therefore *f* the sides of the same IL are given: Wherefore IB is given; and seeing that it is equal *g* to FD, the same FD is also given. But FC is given; therefore the remainder DC *h* is given; and *i* in a given ratio to BD, and therefore *k* BD is given.

a 52. *prop.*
b 36. 1.
c 43. 1.
d 4. *prop.*
e 24. 6.
f 55. *prop.*
g 34. 1.
h 4. *prop.*
i 5. *def.*
k 2. *prop.*

PROP. LIX.



If a given space AB be applied according to a given right line AC, exceeding it by a figure CB given in kind, the breadths of the excesses CE and CF are given.

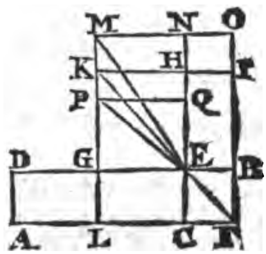
Constr. For DE being divided into two equal parts in G, let there be described on GE the rectiline figure GH, alike and alike posited to CB.

Demonstr. Now seeing that CB is alike to GH, those figures CB and GH * are about one and the same diameter, and GH is given in kind, as is CB. But it is described on the given line GE: Therefore *a* the same GH is also given in magnitude. But AB is given: Therefore AB and GH are given in magnitude. Now those figures AB and GH, are equal to LI, (for AG, LE, and EI, being equal, the Gnomon GFH is equal to AB; and therefore adding GH common to both, LI shall be equal to AB and GH;) therefore LI is given in magnitude; but it is also given in kind, since it is *b* alike to CB. Therefore *c* the sides of the said LI are given, seeing it is equal to GE: Therefore *d* the remainder CF is

a 52. *prop.*
b 24. 6.
c 55. *prop.*
d 4. *prop.*

is given, and in a given ratio *e* to CE. Wherefore *f* CE is given. e 3. def. f. 2. prop.

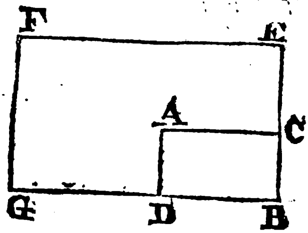
Scholium.



* *EUCLIDE supposeth here that CB and GH are about one and the same diameter, but we shall thus demonstrate it: Let CB and GH be two alike parallelograms disposed as above, that is to say, that the equal angles join together in E, the side CE meets directly with his homologous side EH, and the side BE, his correspondent side EG; and let the diameter FE be drawn, I say that the said diameter FE prolonged, will pass through the point K; that is to say, the parallelograms GH and CB, consist about one and the same diameter. For if it be denied, the diameter EF being produced, will pass above the point K, or below it. Let it in the first place pass above it, and let it cut GK, prolonged in the point M, and through the point M let there be drawn MN, parallel to KH, which shall meet EH, prolonged in the point N, and FB in O.*

Demonstr. Forasmuch as the parallelograms GN and CB are with the parallelogram LO about one and the same diameter, they are *g* alike to one another. Wherefore as FC is to CE, so is EG to GM. In like manner, seeing the parallelograms CB and GH are alike, as FC is to CE, so is EG to GK: Therefore *h* as EG is to GM, so is EG to GK. Wherefore *i* GM and GK are equal, a part to the whole, which is absurd: By the same way of reasoning it may be demonstrated, that the diameter prolonged will not fall below the point K: Therefore the parallelograms CB and GE consist about one and the same diameter. g 24. 6. h 11. 5. i 9. 5.

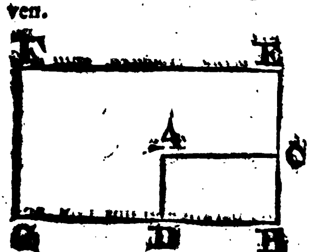
PROP. LX.



If a parallelogram AB, given in kind and in magnitude, be augmented or diminished by a Gnomon CFD, the breadths of the Gnomon (consisting of the lines CE and DG) are given.

Demonstr. For seeing that AB is given, and the Gnomon CFD also given, the whole parallelogram BF is given: But it is also given in kind, seeing it is alike to BA: Therefore *a* the sides of the same BF are given; and therefore each of the lines BE and BG is given. But each of the lines BC and BD is given; therefore each of the remaining lines CE and DG is also given.

a 55. prop.



Constr. Now let the parallelogram BF, given in kind and in magnitude, be diminished by the given Gnomon CFD: I say that each of the lines CE and DG is given.

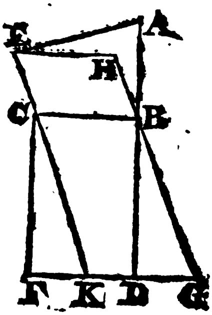
Demonstr. For seeing that BF is given, and the Gnomon CFD given, the remaining figure AB is also given. But it is also given in kind, seeing it is alike to BF: Therefore *b* the sides of the said AB are given, and therefore each of the lines CB and BD is given: But each of the lines BE and BG is given: Therefore also each of the remaining lines CE and DG is given.

b 55. prop.

PROP.

PROP. LXI.

If to one side of a figure ABCE, given in kind, there be applied a parallelogrammic space CD in a given angle BCF, and that the given figure AC hath to the parallelogram CD a given ratio, the parallelogram CD is given in kind.



Constr. For through the point B, let BH be drawn parallel to CE, and through the point E let EH be drawn parallel to CB, and let BC and HB be prolonged to the points K and G.

Demonstr. Forasmuch as the angle BCE is given, and the ratio of EC to CB, *a* the parallelogram CH is given * in kind. But the figure ABCE is also given in kind, and is described on the same line BC, as the parallelogram CH given in kind is: Therefore *b* the ratio of the figure ABCE to the parallelogram CH is given. But by supposition the ratio of the said figure ABCE to the parallelogram CD is also given; and CD is *c* equal to CG: therefore *d* the ratio of CH to CG is given. Wherefore the ratio of the line EC to the line CK is given; (for *e* as CH is to CG, so is EC to CK.) But the ratio of EC to CB is also given: Therefore *f* the ratio of the said CB to CK is given. And seeing that the angle ECB is given, also the following angle BCK *g* is given. But the angle BCF is proposed given; and therefore the remaining angle FCK is given. Also the angle CKF is given, for that *h* it is equal to the angle BCK: Therefore the other angle CFK is given: Wherefore *i* the triangle FCK is given in kind; and therefore the ratio of FC to CK is given. But the ratio of CB to the same CK is also given: Therefore *k* the ratio of FC to CB is given; and the angle BCF is also given. Wherefore the parallelogram CD is given in kind.

a 3. *def.*
b 49. *prop.*
c 36. 1.
d 8. *prop.*
e 1. 6.
f 8. *prop.*
g 13. 1. & 4. *prop.*
h 29. 1.
i 40. *prop.*
k 8. *prop.*

Scholium.

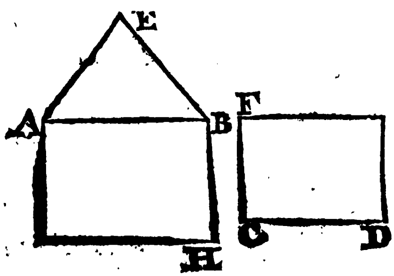
* *Altho' it be manifest that a parallelogram that hath one angle given, and the ratio of the sides about the same angle also given, is given in kind, as* Euclide *doth here de-*

clare,

clare; so it is notwithstanding that the antient interpreter doth thus demonstrate it.

Seeing that in the parallelogram CH the angle ECB is given, the angle CBH is also given; for the right line BC falling on the parallels BH and CB, doth make the two internal angles on the same part equal to two right angles. And therefore seeing that the angle ECB is given, the other angles are given; and seeing that the ratio of EC to CB is given, and that BH is equal to CB, and EH to BC, the ratio of the sides to one another is also given.

PROP. LXII.



If two right lines AB and CD, have to one another a given ratio, and that on one of them AB, there be described a figure AEB, given in kind; but on the other CD, a parallelogrammic space DF in a given angle DCF, and that the figure AEB hath to the parallelogram DF a given ratio, the parallelogram DF is given in kind.

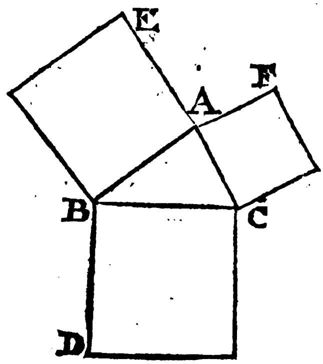
Constr. For on the line AB let there be described the parallelogram AH, alike and alike posited to DF.

Demonstr. Seeing that the ratio of AB to CD is given, and that on those lines are described the rectiline figures
a 50. *prop.* AH and FD, alike and alike posited, *a* the ratio of AH to FD is given. But the ratio of FD to AEB is also
b 8 *prop.* given; Therefore *b* the ratio of AH to AEB is given, But the angle ABH is also given, being equal to the angle FCD, and so the figure AEB is given in kind; and to AB one of the sides thereof, the parallelogram AH is applied in a given angle ABH, and the ratio of the said figure AEB to the said parallelogram AH is gi-
c 61. *prop.* ven: Therefore *c* the parallelogram AH is given in kind; and therefore FD which is alike thereto, is also given in kind.

PROP.

PROP. LXIII.

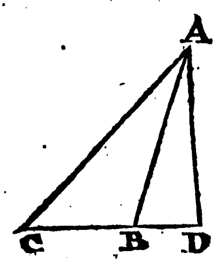
If a triangle ABC be given in kind, the square BE, CD, and CF, which is described on each of the sides, shall have a given ratio to the triangle ABC.



Demonstr. For seeing that on one and the same right line BC, there are described the two rectiline figures ABC and CD, given in kind, *a* the ratio of the same ABC to CD is given; and therefore the ratio of the squares BE and CF, to the triangle ABC is also given. a 40. *prop.*

PROP. LXIV.

If a triangle ABC, hath an obtuse angle ABC given, that space by which the side AC subtending the obtuse angle ABC, is more in power than the sides AB and BC, that comprehend the said angle, shall have a given ratio to the triangle ABC.



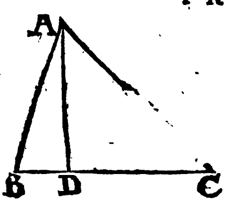
Constr. Let the line CB be prolonged directly, and from the point A let the perpendicular AD be drawn: I say that the space by which the square of the line AC doth exceed the squares of the lines AB and BC, that is to say, *a* the double of the rectangle contained under CB and BD, shall have a given ratio to the triangle ABC. a 12. 2.

Demonstr. For seeing that the angle ABC is given, the angle ABD is also given; but the angle ADB is also given; therefore the other angle BAD is given: Wherefore *b* the triangle ABD is given in kind; therefore *c* the ratio of AD to DB is given. But as AD to DB, so *d* the rectangle of AD and BC is to the rectangle of BC and BD. But the ratio of AD to BD is given: Therefore also is the ratio of the rectangle b 40. *prop.* c 3. *def.* d 1. 6.

of

of AD and BC to the rectangle of BC and BD given: Wherefore the ratio of the double of the said rectangle BC and BD to the rectangle of AD and BC is also given. But the said rectangle of AD and BC hath also a given ratio to the triangle ABC (to wit, double ratio; for
e 41. 1. the rectangle is *e* double to the triangle) therefore the
f 8. prop. ratio of the double of the rectangle of BC and BD *f* to the triangle ABC is given. But the same double of the rectangle of CB and BD is that space by which the square of the line AC doth exceed the squares of the lines AB and BC: Therefore the same space hath a given ratio to the triangle ABC.

PROP. LXV.



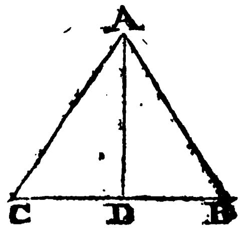
If a triangle ABC, hath one acute angle ADB given, that space, by which the side subtending the said acute angle is less in power than the sides comprehending the same acute angle, shall have a given ratio to the triangle.

Constr. From the point A let there be drawn the line AD, perpendicular to BC: I say, that space by which the square of the line AB is less than the squares of the
a 13. 2. lines AC and CB, that is to say, *a* the double of the rectangle of BC and CD, hath a given ratio to the triangle ABC.

Demonstr. For seeing that the angle C is given, and the angle ADC also given, the other angle DAC is
b 40. prop. given: Wherefore the triangle *b* ADC is given in kind; and therefore the ratio of AD to DC is given, and con-
c 1. 6. sequently also *c* that of the rectangle of BC and CD to the rectangle of BC and AD: Therefore the ratio of the double of the rectangle of BC and CD to the rectangle of BC and AD is given. But the ratio of the same rectangle of BC and AD to the triangle ABC is given (for
d 41. 1. *d* the rectangle is double to the triangle:) Therefore *e*
e 8. prop. the ratio of the double of the rectangle of BC and CD to the triangle ABC is given. And seeing that the same double of the rectangle of BC and CD is that whereby the square of the line AB is less than the squares of the lines AC and BC, that space by which the square of the line AB is less than the squares of the lines AC and BC, shall have a given ratio to the triangle ABC.

PROP.

PROP. LXVI.



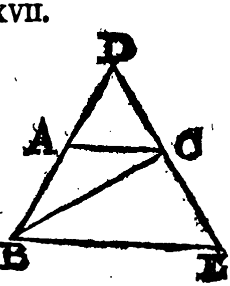
If a triangle ACB, hath one angle B given, the rectangle made of the lines AB and BC, containing the same angle, shall have a given ratio to the triangle.

Constr. For from the point A let AD be drawn perpendicular to CB.

Demonstr. Therefore seeing that the angle B is given, and also the angle ADB; the other angle BAD is likewise given. Wherefore the triangle ADB *a* is given in kind; and consequently the ratio of AB to AD is given. But as AB is to AD, *b* so the rectangle of AC and CB is to the rectangle of CB and AD: Therefore the ratio of the rectangle of AC and CB to the rectangle of CB and AD is given. But the ratio of the said rectangle of CB and AD to the triangle ACB is also given; (for that it is double ratio, the rectangle being double *c* to the triangle:) Therefore *d* the ratio of the rectangle of AC and CB to the triangle ABD is given.

a 40. *prop.*
b 1. 6.
c 41. *prop.*
d 8. *prop.*

PROP. LXVII.



If a triangle ABC hath one angle BAC given, that space by which the square of the line compounded of the two sides BA and AC, that contain the same given angle BAC doth exceed the square of the other side, shall have a given ratio to the triangle ABC.

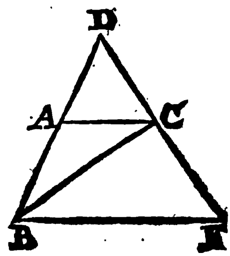
Constr. For let BA be prolonged in such sort as that AD may be equal to AC, then having drawn DCE infinitely, from the point B let BE be drawn parallel to AC, meeting the said DE in the point E.

Demonstr. Forasmuch as AD is equal to AC, *a* DB is equal to BE; (for the two triangles ADC and BDE are alike) and from the top B is drawn to the base DE, the right line BC: Therefore * the rectangle of DC and CE, with the square of BC, is equal to the square of BD; but the

a 4. 6. & 14. 5.

the same BD is compounded of BA and AC; therefore the square of the compound of AB and AC is greater than the square of BC, of the rectangle of DC and CE.

Now I say that the rectangle of DC and CE hath a given ratio to the triangle ABC: Forasmuch as the angle BAC is given, the angle DAC is also given. But each of the angles ADC and ACD is given, it being the half
b 40. prop. of the angle BAC which is given. Therefore *b* the triangle ADC is given in kind; and therefore the ratio of
c 50. prop. DA to DC is given. Therefore *c* the ratio of the square of the said DA to the square of DC is also given. And
d 2. 6. seeing that as BA is to AD, *d* so is EC to CD, and also as
e 1. 6. BA is to AD, *e* so is the rectangle of BA and AD to the square of AD; and as EC is to
f 1. 6. CD, *f* so also is the rectangle of EC and CD to the square of CD; by permutation, as the rectangle of BA and AD is to the rectangle of EC and CD, so is the square of AD to the square of DC. But the ratio of the said square of AD to the square of DC is given: Therefore the ratio of the rectangle of BA and AD to the rectangle of EC and CD is also given. But AD is equal to AC: Therefore the ratio of the rectangle of BA and AC to the rectangle of EC and CD is given. But the ratio of the rectangle of BA and AC to the triangle
g 66. prop. ABC *g* is given, because the angle BAC is given:
h 8. prop. Therefore *h* the ratio of the rectangle EC and CD to the triangle ABC is given. But the rectangle of EC and CD is that whereof the square of the line compounded of BA and AC is greater than the square of BC: Therefore that space by which the square of the line compounded of BA and AC is greater than the square of BC, shall have a given ratio to the triangle ABC.



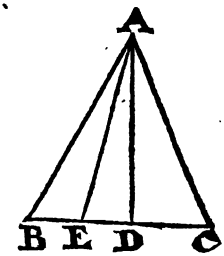
Scholium.

* *EUCLIDE supposeth in this place, that when in an Isosceles triangle a right line is drawn from the top to the base, the square of that line, with the rectangle contained under the segments of the bases, is equal to the square of either of the other legs, which the antient interpreter doth thus demonstrate.*

Constr.

Constr. Let ABC be an Isosceles triangle, whose legs are AB and AC; and from the top A let AD be drawn to the base BC: I say, that the square of AD with the rectangle of BD and DC, is equal to the square of either of the legs AB or AC.

Demonstr. Now the line AD is perpendicular to BD, or not: Let it in the first place be perpendicular: Therefore it will cut the base BC into two equal parts in the point D; and therefore the rectangle contained under BD and DC is equal to the square of the said BD, and adding to them the common square of AD, the rectangle of BD and DC with the square of AD, shall be equal to the squares of DB and AD. But to those squares of AD and DB *i* the square of AB is equal: Therefore the square of AB is equal to the rectangle of BD and DC, and the square of AD together. i 47. 1.



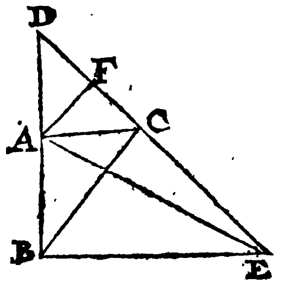
Now suppose AD not to be perpendicular, but that from the point A there doth fall on BC the perpendicular AE, that being so, BC shall be cut into two parts equally in the point E, and unequally in D. Wherefore the rectangle of BD and DC, with the square of DE, *k* is equal to the square of BE; and adding the common square of AE, the rectangle of BD and DC, with the squares of DE and AE, shall be equal to the squares of BE and AE. But *l* the square of AD is equal to the two squares of DE and AE: Therefore the rectangle of BD and DC, with the square of AD, is equal to the squares of BE and AE. But to these squares of BE and AE the square of AB is equal: Therefore the square of AD, with the rectangle of BD and DC, is equal to the square of AB. k 5. 2. l 47. 1.

OTHERWISE.

Constr. Having done, as in the foregoing Demonstration, from the point A, let AF be drawn perpendicular to CD, and let AE be drawn.

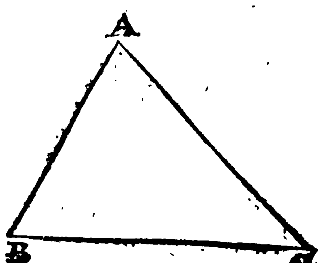
Demonstr. Forasmuch as the angle BAC is given, the half thereof ACF shall be also given. But the angle AFC is given; and therefore the triangle AFC is given in kind: Therefore the ratio of AF to FC is given. But the ratio of CD to the same FC is also given, seeing that

m 8. prop. that CD is double to FC: Therefore m the ratio of CD to AF is given; and therefore also the ratio of the rectangle of CD and EC, to the rectangle of AF and
n 1. 6. EC, is given; (for it is the same ratio n as that of CD to AF.) But the ratio of the rectangle of AF and FC to the triangle ACE is given; seeing it is double
o 41. 1. o to the same triangle. Therefore the ratio of the rectangle of CD and CE to the triangle ACE is also given. But the triangle ACE is equal to
p 37. 1. the triangle ABC p, they being both constituted on one and the same base AC, and between the



q 8. prop. same parallels AC and BE: Therefore q the ratio of the rectangle of CE and CD to the triangle ABC is given. But the said rectangle of CE and CD is the space by which the square of the line compounded of AB and AC, is greater than the square of BC: Therefore that space by which the square of the line compounded of AB and AC is greater than the square of BC, hath a given ratio to the triangle ABC.

OTHERWISE.

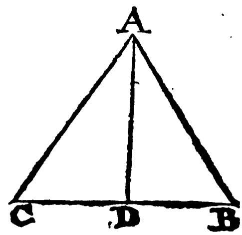


For the given angle A is either a right, acute, or obtuse angle: Let it in the first place be supposed a right angle: Therefore the square of the line compounded of BAC, is greater than the square of BC, by twice the rectangle of BA and AC; (seeing
r 47. 1. that r the square of BC is equal to the squares of BA and AC; and the square of the line compounded of
s 4. 2. BAC s is equal to those two squares of BA and AC, and twice the rectangle of the said BA and AC:) Wherefore the ratio of double the rectangle of BA and AC to the triangle ABC is given.

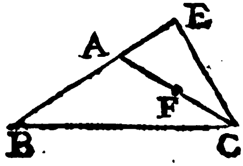
Const,

Constr. Now let the angle C be supposed acute, and from the point A let there be drawn on CB the perpendicular AD.

Demonstr. Forasmuch as the triangle CAB is an Oxigonium triangle, and the perpendicular AD being drawn, the square of CA and CB are equal *t* to the square of AB with twice the rectangle of CB and CD; adding therefore the common double rectangle of CA and CB, the squares of CA and CB, with the double rectangle of the said CA and CB, that is to say, *u* the alone square of the line compounded of ACB, are equal to the square of AB, with the double of the rectangle of CD and CB, and over and above the double of the rectangle of AC and CB, that is to say, the double of the rectangle contained under the compound line of ACD and CB (for the rectangle of ACD and CB is *x* equal to the rectangles of AC and CB, and of CD and CB:) Therefore the square of the line compounded of ACB is greater than the square of AC, by double the rectangle of ACD and CB. And seeing that the angle ACB is given, and the angle BDA also given, the other angle CAD is given: Therefore *y* the triangle CAD is given in kind, and therefore the ratio of CD to CA is given, and by consequence the ratio of the line compounded of ACD to CA *z* is also given. Wherefore the ratio of the rectangle of those lines compounded of ACD and CB *a* to the rectangle of AC and CB is also given. But the ratio of the said rectangle of AC and CB to the triangle CAB *b* is given, seeing the angle C is given; therefore the ratio of double the rectangle of the line compounded of ACD and CB to the triangle CAB is given.



t 13. 5.
u 4. 2.
x 1. 2.
y 40. prop.
z 6. prop.
a 1. 6.
b 66. prop.



Lastly, let the angle BAC be supposed to be obtuse, and having prolonged BA, from the point C, let the perpendicular CE be drawn on the said line BA prolonged; and let AF be proposed to be equal to AE.

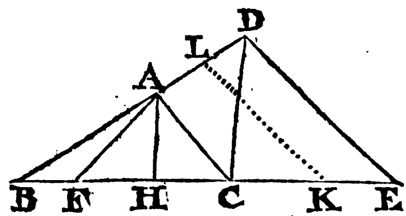
Demonstr. Forasmuch as the angle BAC is obtuse, and the perpendicular CE being drawn, the squares of AB and AC, and the double of the rectangle under BA and
c 12. 2. AE, or AF, are all alike equal *c* to the square of BC, and adding the common double rectangle of BA and AC, the squares of the said AB and AC, with the double of the rectangle of the same AB and AC, that is to say,
d 4. 2. *d* the square of the line compounded of BAC and the double of the rectangle of BA and AF are together equal to the square of BC, with the double of the rectangle of BA and AC. Let the common double of the rectangle of BA and AF be taken away, and there will remain the square of the line compounded of BAC, equal to the square of BC, with the rectangle of AB and CF;
e 1. 2. (for the rectangle of AB and AC is equal *e* to the two rectangles of AB and AE, and of AB and CF :) Therefore the square of the line compounded of BAC is greater than the square of BC by the double of the rectangle of AB and CF. And forasmuch as the angle BAC is
f 13. 11. given, the angle CAB *f* is given. But the angle ABC is also given; therefore the other angle ACE is given:
g 40. *prop.* Wherefore *g* the triangle ACE is given in kind, and therefore the ratio of CA to AE, that is to say, to AF
h 5. *prop.* is given. Therefore *h* the ratio of the said CA to FG is
i also given. But the ratio of the same CA to CE is
8. *prop.* given; therefore *i* the ratio of CE to CF is also given. Wherefore the ratio of the rectangle of EC and AB to the rectangle of FC and AB is given; (for the rectangle
k 2. 6. is to the rectangle *k* as CE is to CF) and also that of the rectangle of AC and AB to the rectangle of EC and
l 8. *prop.* AB. Therefore *l* the ratio of the rectangle of FC and AB to the rectangle of AC and AB is given. But the ratio
m 66. *prop.* of the rectangle of AC and AB to the triangle ABC *m* is given: Therefore also the ratio of the double of the rectangle of FC and AB, to the triangle ABC is given. But the same double of the rectangle of FC and AB, is that whereby the square of the line compounded of BAC is

is greater than the square of BC, wherefore that space by which the square of the line compounded of BAC is greater than the square of BC, hath a given ratio to the triangle ABC.

OTHERWISE.

Constr. Let the line BA be prolonged to the point D, in such sort as AD may be equal to AC, and let CD be drawn.

Demonstr. Forasmuch as the angle BAC is given, each of the angles ADC and ACD, which is the half thereof, shall be also given; and therefore the other angle DAC is also given: Therefore *n* the triangle ACD is given in kind. Wherefore the ratio of AC to CD is given. And n 40. *prop.*

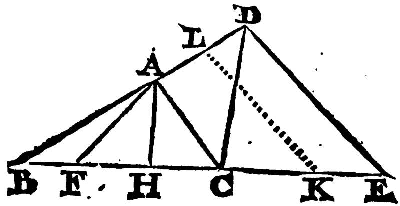


forasmuch as the angle ADC is given: Let each of the angles DEC and AFC be made equal to the said ADC: Therefore seeing that the angle BDC is equal to the angle DEC, and the angle DBE is common to the triangles DBE and DBC, the other angle BDE is equal to the other angle BCD; and therefore the triangle BDE is equiangled to the triangle BDC. Therefore *o* as EB is to BD, so is BD to CB: Wherefore the rectangle of EB and CB, that is to say, *p* the rectangle of EC and CB, *q* with the square of CB is equal, *r* to the square of BD, that is to say, to the square of the line compounded of BAC; for AD is equal to AC; and therefore the rectangle of EC and CB with the square of CB, that is to say, the square of the line compounded of BAC is greater than the square of the rectangle of BC and CE: I say therefore that the ratio of the said rectangle of BC and CE to the triangle ABC is given. Forasmuch as the angle BDE is equal to the angle BCD, and the angle ADC equal to the angle ACD, the other angle CDE is equal to the other angle ACB: But the angle DEC is also equal to the angle AFC; therefore the remaining angle CAF is equal to the remaining angle DCE. Wherefore the triangle AFC is equiangled to the triangle DCE; and therefore *s* as CA is to AF, so is CD to CE; and by permutation, as AC is to CD, so is AF to CE. But the ratio of AC to CD is

o 4. 6.
p 5. 2.
q 5. 2.
r 17. 6.
s 4. 6.

 is

is given: Therefore also the ratio of AF to CE is given. From the point A let AH be drawn perpendicular to BC: Forasmuch as the angle AFC is given, and the angle AHF also given, the third angle HAF is given:
t 40. prop. Wherefore *t* the triangle AHF is given in kind; and by consequence the ratio of AF to AH is given. But the ratio of AF to CE is also given:



u 8. prop. Therefore *u* the ratio of AH to CE is given; and therefore the ratio of the rectangle of AH and BC
x 1. 6. *x* to the rectangle of BC and CE is also given. But the ratio of the rectangle of AH and BC, to the triangle
y 41. 1. ABC is likewise given; (for the rectangle *y* is double to the triangle) and the rectangle of BC and CE is that whereby the square of the line compounded of BAC is greater than the square of BC. Therefore that space by which the square of the line compounded of BAC is greater than the square of BC has a given ratio to the triangle ADC.

Scholium.

† *The antient Interpreter pretending to shew the construction of the angle DEB equal to the angle ADC, saith that on the line BD and in the point D, the angle BDE ought to be made equal to the angle BCD, and that the right lines BC and DE be drawn until they intersect in E, in such sort as he supposeth the angle BCD, to be given; but it is not.*

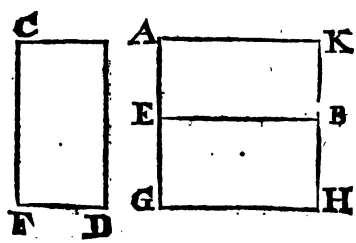
The same Interpreter afterwards shews how there may universally from a given point be drawn a right line, given in position to a right line, making an angle equal to a given angle. But we will also reject this way, seeing we have elsewhere shewn another more brief and easy. For example, if we would from the point D draw to the line BC given in position a right line, making an angle equal to a given angle ADC, as is here required, we have no more to do but to assume the point K in the said line BC, and there make the triangle CKL equal to the given angle ADC: If the line KL doth meet with the point D, it shall be the line required. But if it meet not with it, from the point D let there be drawn the line DE parallel

parallel to the said KL, cutting BC prolonged in E, and the angle DEC shall be equal to the given angle ADC, for on the two parallel lines, LK and DE, there doth fall the line BE; and therefore the angle DEC z *is equal to the angle LKC, which hath been made equal to the given angle ADC; and by consequence the same angle DEC is also equal to ADC.* z 29. 1.

PROP. LXVIII.

If two parallelograms AB and CD have to one another a given ratio, and that a side hath also a given ratio to a side, the other side shall have likewise a given ratio to the other side.

Constr. Let the ratio of BE to FD be given: I say the ratio of AE to FC is also given. For to the right line EB let there be applied the parallelogram EH, equal to the parallelogram CD, and constituted in such sort as AE and EG may make one right line: † Therefore KB and BH will also make one right line.



Demonstr. Forasmuch as the ratio of AB to CD is given, and that EH is equal to the said CD; the ratio of *a* AB to EH is given; and therefore the ratio of AE to EG is also given. Seeing therefore that EH is equal and equiangled to CD, as *b* EB is to FD, so is FC to EG. But the ratio of EB to FD is given: Therefore also the ratio of FC to EG is given. But the ratio of AE to the same EG is also given: Therefore the ratio of AE to FC is given. a 1. 6. b 14. 6.

Scholium.

† *EUCLIDE having posited AE and EG directly in one right line, presently concludeth that KB and BH shall also make a right line; but we shall demonstrate it thus. Seeing the lines AE and EG are posited directly, the angles AEB and BEG* c *are equal to two right angles; and seeing that AB is a parallelogram, the lines AK and EB are parallels, on which the line AE doth fall; and therefore the two internal angles A and BEA* d *are also equal to* c 13. 1. d 29. 1.

two right angles, and taking away the common angle BEA, there will remain the angle A, equal to the angle BEG; and consequently their opposite angles EBK and H are also equal to one another. Again, seeing that BG is a parallelogram, the two lines BE and HG are parallels, on which BH doth fall; and therefore the two internal angles H

d 29. 1. *and EBH d are equal to two right angles. But it hath been demonstrated that H is equal to EBK: Therefore the two angles EBK and EBH are also equal to two right*

e 14. 1. *angles; and therefore e the two lines KB and BH do meet directly according to EUCLIDE.*

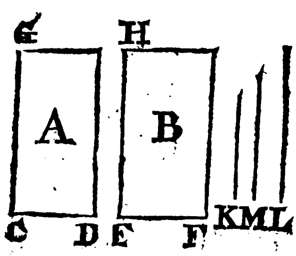
OTHERWISE.

Constr. Let the given right line K be exposed, and seeing that the ratio of A to B is given, let the same be made of K to L; therefore the ratio of K to L is also given.

f 2. *prop.* *Demonstr.* But K is given; therefore *f* L is also given. Again, seeing that the ratio of CD to EF is given, let the same be made of K to M: Therefore the ratio of K

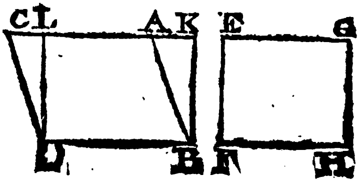
g 1. *prop.* to M is given. But K is given, therefore *g* M is also given; and therefore the ratio of L to M is given. Now seeing that A is equiangled

h 23. 6. to B, *h* the ratio of the said A to B is compounded of that of the sides, that is to say, of CD to EF, and of CG to EH. But also the ratio of K to L is compounded of K to M, and of M to L; therefore the ratio compounded of CD to EF, and of CG to EH, is the same with that which is compounded of K to M, and of M to L (the ratio of K to L being the same as of A to B:) But the ratio of CD to EF is the same as of K to M: Therefore the other ratio of CG to EH is also the same as of M to L. But the said ratio of M to L is given: Therefore also the ratio of CG to EH is given.



PROP.

PROP. LXIX.



If two parallelograms, CB and EH, having the angles D and F given, and that a side hath also a given ratio to a side; in like manner the other side shall have a given ratio to the other side.

Constr. Let the ratio of BD to FH be also given : I say that the ratio of AB to EF is given. For if CB be equiangled to HE, it is manifest by the precedent Proposition; but if it be not equiangled thereto, let the right line DB be constituted, and in the given point B therein, let the angle DBK be made equal to the angle EFH, and finish the parallelogram DK.

Demonstr. Forasmuch as each of the angles BKL and BAK is given, † the other angle KBA is given : Wherefore the triangle *a* ABK is given in kind; and therefore the ratio of AB to BK is given. But the ratio of CB to EH is supposed to be given, and *b* CB is equal to DK; therefore the ratio of DK to EH is given; and seeing that DK is equiangled to EH, and the ratio of the said DK to EH is given, as also that of DB to FH, *c* the ratio of BK to FE is given. But the ratio of the said BK to BA is also given: Therefore *d* the ratio of AB to FE is given.

a 40. *prop.* b 35. *prop.* c 68. *prop.* d 29. 1.

Scholium.

† *EUCLIDE supposeth here, that a parallelogram having one angle given, all the other angles are also given, and as well the antient Interpreters as others, do give the reasons why, the angle F being given, the other angle E shall be also given, it being the remainder of two right angles, for that on the parallel lines EG and FH there doth fall the line EF, which makes* e *the two internal angles (of the same part) F and G, equal to two right angles. But to those angles* f *the opposite angles G and H are equal, and therefore they are also given.*

e 29. 1. f 34. 1.

From whence it follows that the angles BDC and F being given by supposition, all the other angles of the two parallelograms CB and EH, are also given : Therefore the angle

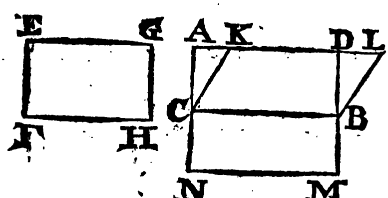
angle DBK having been made equal to the angle F, the angle K shall be equal to the angle E, and given as that is: But the angle BAL, which is opposite to the given angle BDC, is also given; and therefore BAK, which is the remainder of two right angles, shall be also given; in such sort as in the triangle ABK, the two angles BAK and BKA are given, as EUCLIDE doth declare in this place.

PROP. LXX.

If of two parallelograms AB and EH, the sides about the equal angles, or about the unequal angles (yet nevertheless given angles) have to one another a given ratio, to wit (AC to EF, and CB to FH) also the same parallelograms AB and EH shall have to one another a given ratio.

Constr. For let AB be prolonged to EH, and on the right line CB let the parallelogram CM be applied equal to the parallelogram EH, in such sort as AC may be direct to CN; that is to say, that AC and CN make
a *sch.* 68. *prop.* one right line; and by consequence DB shall be *a* directly with BM.

Demonstr. Forasmuch then as CM is equiangled and equal to EH, the sides about the equal angles shall be
b 14. 6. reciprocally *b* proportional: Wherefore as BC is to HF, so is FE to NC. But the ratio of BC to HF is given: Therefore the ratio of FE to NC is also given. But the ratio of AC to the same EF is



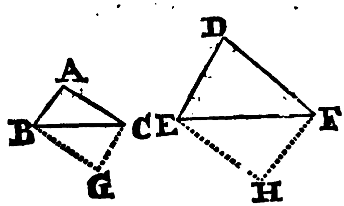
c 8. *prop.* given: Therefore *c* the ratio of AC to NC is also given. Wherefore the ratio of AB to CM is given; (for it is
d 1. 6. the same *d* as of AC to CN.) But CM is equal to EH: Therefore the ratio of AB to EH is given.

Constr. Now suppose AB not to be equiangled to EH, and on the right line CB, and in the given point C therein: Let there be constituted the angle BCK, equal to the given angle F, and so finish the parallelogram CL.

Demonstr. Forasmuch as the angle ACB is given, and the angle BCK also given, the remaining angle ACK

ACK is given: Therefore the triangle ACK e is given in kind: and therefore the ratio of AC to CK is given: But the ratio of AC to EF is also given: Therefore the ratio of CK to EF is given. But the ratio of BC to HF is also given, and the angle BCK is equal to the angle F; therefore (by the first part of this proposition) the ratio of CL to EH is given. But to the said CL, AB is equal: Therefore the ratio of AB to EH is given. e 40. prop.

PROP. LXXI.

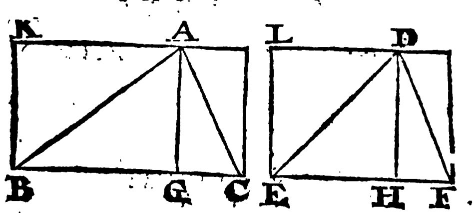


If of two triangles ABC and DEF, the sides about the equal angles A and D, or else about the unequal angles (yet nevertheless given angles) have to one another a given ratio (to wit, AB to DE and AC to DF) the same triangles shall have also to one another a given ratio ABC to DEF.

Constr. Let the parallelograms AG and DH be finished.

Demonstr. Seeing that the two parallelograms AG and DH, have the sides about the equal angles A and D, or else about the unequal angles (nevertheless given) in a given ratio to one another, the ratio *a* of the parallelogram AG to the parallelogram DH is given. But the triangle ABC is the half of the parallelogram AG *b* and the triangle DEF the half of the parallelogram DH. Therefore the ratio of the triangle ABC to the triangle DEF is given. a 70. prop. b 34. prop.

PROP. LXXII.



If of two triangles ABC and DEF, the bases BC and EF, are in a given ratio, BC to EF, and that from the angles A and D, there be drawn to those bases the right lines AG and

and DH, making the equal angles AGC and DHF, or else unequal (yet neverthelesss given) which shall have to one another given ratio's AG to DH, those triangles ABC and DEF shall have also a given ratio to one another, to wit, ABC to DEF.

Constr. For let the parallelograms KC and LF be finished.

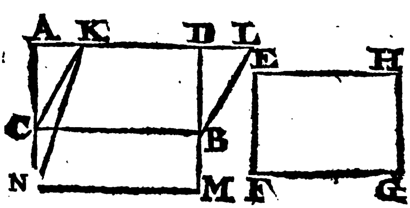
Demonstr. Forasmuch as the angles AGC and DHF are equal, or unequal (yet given) and that the angle

a 29. 1. AGC *a* is equal to the angle KBC, and also the angle DHF equal to the angle LEF, the angles at the points B and E are equal, or else unequal (yet given,) and because the ratio of AG to DH is given, and AG is equal to KB, and DH is equal to LE, therefore the ratio of KB to LE is given. But the ratio of BC to EF is also given, and the angles at the points B and E are equal, or else

b 70. prop. unequal (yet given:) Therefore *b* the ratio of the parallelogram KC to the parallelogram LF is given; and therefore the ratio of the triangle ABC to the triangle

c 41. 1. DEF is given, seeing those triangles *c* are the one half of the parallelograms.

PROP. LXXIII.



If of two parallelograms AB and EG, the sides about the equal angles C and F, or else about the unequal angles (but neverthelesss given) are in such sort to one another, that as the side CB of the first, is to the side FG of the second; so the other side EF of the second, is to some other right line CN. But that the other side AC, hath also to the same right line CN a given ratio, those parallelograms will have also to one another a given ratio AB to EG.

Constr. For in the first place, let the parallelogram AB be equiangled to EG, and having placed CN directly to AC: Let the parallelogram CM be finished.

Demonstr. Forasmuch then as CB or NM its equal, is to FG, so is EF to CN, and that the angles N and F are equal (for N is equal to the angle ACB, which is

a 14. 6. put equal to F) the parallelograms CM and EG *a* are

equal: But as AC to CN, so *b* the parallelogram AB is to the parallelogram CM or EG: Therefore seeing that the ratio of AC to CN is given, the ratio of AB to EG is also given. b 1. 6.

Constr. 2. Now suppose the parallelogram AB not to be equiangled to the parallelogram EG, and let there be constituted at the given point C in the line CB, the angle BCK, equal to the angle EFG, and so finish the parallelogram CL.

Demonstr. 2. Seeing that each of the angles ACB and KCB is given, the remaining angle ACK is also given. But *c* the angle CAK is given, as also the remaining angle AKC: Therefore *d* the triangle ACK is given in kind; and therefore the ratio of AC to CK is given. But the ratio of the same AC to CN is also given: Therefore *e* the ratio of CK to CN is given. And seeing that as CB is to FG; so is EF to the right line CN, to which the other side KC hath a given ratio, and that the angle BCK is equal to the angle F, the ratio of the parallelogram CL to the parallelogram EG is given (by the first part of this proposition) but the parallelogram CL is equal to the parallelogram AB: Therefore the ratio of the parallelogram AB to the parallelogram EG is given. c *sch.* 69. *prop.* d 40. *prop.* e 8. *prop.*

PROP. LXXIV.

If two parallelograms (as in the former figure) AB and EG, in equal angles C and F, or else in unequal angles (yet nevertheless given angles) have a given ratio to one another, as one side CB of the first shall be to one side FG of the second, so the other side EF of the second, shall be to that to the which the other side AC of the first hath a given ratio. (See the foregoing Scheme.)

Constr. For either AB is equiangled or not; suppose it in the first place to be equiangled, and to the right line BC let there be applied the parallelogram CM, equal to the parallelogram EG, and so posited, as that AC and CN may be direct: Therefore *a* DB and BM shall be also direct (that is, as one right line.) a *sch.* 68. *prop.*

Demonstr. Seeing that the ratio of AB to EG is given, and that CM is equal to EG, the ratio of AB to CM is also given; and therefore the ratio of AC to CN is given (seeing AB is to CM, *b* as AC is to CN;) and for that CM is equal and equiangled to EG, the sides about the equal angles of the parallelograms CM and EG, *c* are reciprocally proportional; and therefore as CB is b 1. 6. c 14. 6.

to

to FG, so is EF to CN. But the ratio of AC to CN is given: Therefore as CB is to FG, so is EF to that to which AC hath a given ratio.

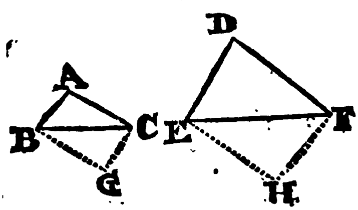
Constr. 2. Now suppose AB not to be equiangled to EG, and in the given point C of the line CB, let there be constituted the angle BCK equal to the angle EFG, and finish the parallelogram CL.

Demonstr. 2. Seeing then that the ratio of AB to EG is given, and *d* that AB is equal to CL, also the ratio of CL to EG is given, and the angle BCK is equal to the angle F, and therefore CL *e* is equiangled to EG: Therefore (by the first part of this proposition) as CB is to FG, so is EF to that to the which CK hath a given ratio. But the ratio of AC to CK is given; (as appears by what hath been demonstrated in the latter part of the precedent proposition.) Therefore as CB is to FG, so is EF to that to which AC hath a given ratio.

d 36. 1.

e sch. 69. prop.

PROP. LXXV.



If two triangles ABC and DEF, in equal angles A and D, or else unequal (yet neverthelefs given) have to one another a given ratio, as the side AB of the first, shall be to the side DE of the second, so the other side DF of the second, shall be to that right line to the which the other side AC of the first hath a given ratio.

Constr. For let the parallelograms AG and DH be finished.

Demonstr. Forasmuch as the ratio of the triangle ABC to the triangle DEF is given, also the ratio of the parallelogram AG to the parallelogram DH is given.

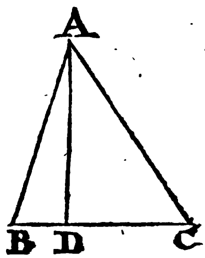
Seeing therefore that the two parallelograms AG and DH in equal angles, or unequal angles (neverthelefs given) have to one another a given ratio; as *a* AB is to DE, so is DF to that to which AC hath a given ratio.

a 74. prop.

PROP.

PROP. LXXVI.

If from the top A of a triangle ABC, given in kind, there be drawn to the base BC, a perpendicular line AD, that line AD shall have to the base BC a given ratio.



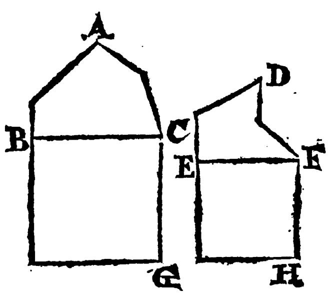
Demonstr. For seeing that the triangle ABC is given in kind, the ratio of AB to BC is given; and the angle B is also given. But the angle ADB is given; therefore the other angle BAD is given. Wherefore *a* the triangle ABD is given in kind; and therefore the ratio of AB to AD is given. But the ratio of AB to BC is given: Therefore *b* the ratio of AD to BC is given.

a 40. *prop.*
b 8. *prop.*

PROP. LXXVII.

If two figures ABC and DEF, given in kind, have to one another a given ratio, the ratio also shall be given of which you please of the sides of one of the figures, to which you please of the sides of the other figure.

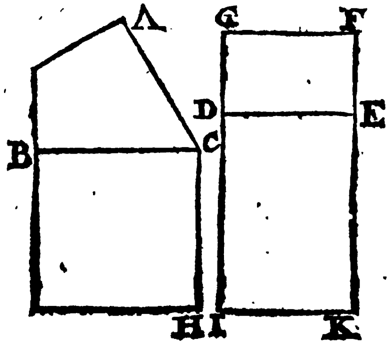
Constr. For on the right lines BC and EF, let there be described the squares BG and EH.



Demonstr. Forasmuch as on one and the same right line BC, are described two figures ABC and BG given in kind, *a* the ratio of the said ABC to BG is given. In like manner the ratio of DEF to EH is given; and seeing that the ratio of ABC to DEF is given, and also that of the same figure ABC to BG; and again the ratio of DEF to EH: *b* the ratio of BG to EH is given; and therefore the ratio of BC to EF is also given.

a 49. *prop.*
b 8. *prop.*

PROP. LXXVIII.



If a given figure ABC, hath a given ratio to some rectangled figure DF, and that one side BC hath a given ratio to one side DE, the rectangled figure DF is given in kind.

Constr. For on the right line BC let the square BH be described, and to the right line DE, let the parallelogram DK be applied equal to BH, in such a manner, as that GD and DI may be placed directly, *a* and by consequence FE and EK also directly.

a *sch.* 68. *prop.*

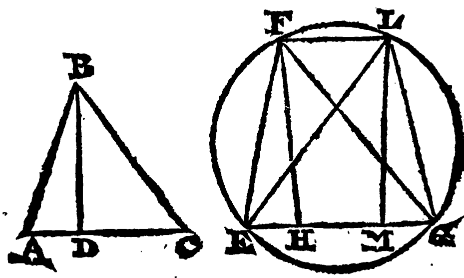
Demonstr. Therefore seeing that on one and the same right line BC are described the two rectiline figures ABC and BH, given in kind, *b* the ratio of ABC to BH is given. But the ratio of the said ABC to DF is also given: Therefore *c* the ratio of BH to DF is given. But BH is equal to DK: Therefore the ratio of DK to DF is also given. And seeing that BH is equal and equiangled to DK, both the one and the other being rectangles, *d* the sides of those figures are reciprocally proportional; and as BC is to DE, so is DI to CH. But by supposition, the ratio of BC to DE is given; therefore also the ratio of DI to CH is given; but the ratio of DI to DG is also given: (for DI is to DG *e* as DK to DF:) Therefore *f* the ratio of DG to CH is given. But CH is equal to BC, seeing that BH is a square; therefore the ratio of the same BC to DG is given: But the ratio of the same BC to DE is also given: therefore the ratio of DE to DG is given, and the angle at D is a right angle: Therefore *g* DF is given in kind.

b 49. *prop.* c 8. *prop.* d 14. 6. e 1. 6. f 8. *prop.* g *sch.* 61. *prop.*

PROP. LXXIX.

If two triangles ABC and EFG, have an angle B equal to an angle F. And from the equal angles B and F there be drawn perpendiculars BD and FH, to the bases AC and EG;

EG; and that as the base AC of the first triangle ABC, is to the perpendicular BD, so also the base EG of the other triangle EFG, is to the perpendicular FH, those triangles ABC and EFG are equiangled.



Constr. For about the triangle EFG let there be described the circle EFLG, then on the right line EG, and in the point E given therein, let there be made the angle GEL, equal to the angle C, and let FL and LG be drawn, and the perpendicular LM.

Demonstr. Seeing then that the angle GEL is equal to the angle C, and the angle ELG is equal to the angle EFG, *a* they being in one and the same segment of the circle; the third angle EGL is equal to the third angle A: Wherefore the triangle ABC is alike to the triangle ELG, and the perpendiculars BD and LM are drawn: Therefore † as AC is to BD, so is EG to LM; but by supposition as AC is to BD; so is EG to FH: Therefore *b* LM is equal to FH. But the said LM is *c* parallel to FH: Therefore *d* FL is also parallel to EG; and therefore the angle FLE *e* is equal to the angle LEG. But the angle C is also equal to the said angle LEG, and the angle FLE to the angle FGE *f*: Therefore also the angle C is equal to the angle FGE. But by supposition the angle ABC is equal to the angle EFG: Therefore the third angle BAC is equal to the third angle FEG: Wherefore the triangle ABC is equiangled to the triangle EFG.

a 21. 3.
b 7. 5.
c 28. 1.
d 33. 1.
e 29. 1.
f 21. 3.

Scholium.

† *Now that as AC is to BD, so EG is to LM, it is by some thus demonstrated. Forasmuch as the angle C is equal to the angle GEL, and the angle BDC to the angle LME, each being a right angle, the other angle CBD is equal*

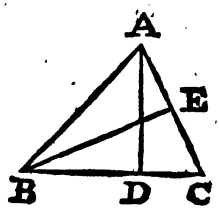
g 4. 6. *equal to the other angle* ELM: *Therefore* g *as* EM *is to* ML, *so is* CD *to* DB. *Again, seeing the angle* ABC *is equal to the angle* ELG, *and the angle* CBD *to the angle* ELM, *the remaining angle* ABD *is equal to the remaining angle* MLG; *but the angle* ADB *is also equal to the angle* LMG; *and therefore the third angle* A *is equal to the third angle*
h 4. 6. LGM: *Therefore* h *as* AD *is to* DB, *so is* GM *to* ML. *But it hath been demonstrated, that as* CD *is to* DB, *so is* EM
i 14. 5. *to* ML: *Therefore* i *as* AC *is to* BD, *so is* EG *to* LM.

PROP. LXXX.

If a triangle ABC *hath one angle* A *given, and that the rectangle contained under the sides* AB *and* AC, *comprising the given angle* A, *hath a given ratio to the square of the other side* BC, *the triangle* ABC *is given in kind.*

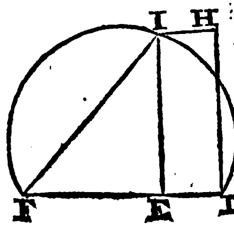
Constr. For from the points A and B, let there be drawn the perpendiculars AD and BE.

Demonstr. Forasmuch as the angle BAE is given, and also the angle AEB, the triangle ABE is given in
a 40. *prop.* *a* kind; and therefore the ratio of AB to BE is given: Therefore the ratio of the rectangle of AB and AC to the rectangle of BE and AC is also given (for it is the
b 1. 6. same ratio *b* as of AB to BE.) But the rectangle of AC and BE is equal to the rectangle of BC and AD; for that each of
c 41. 1. those rectangles is *c* double to the triangle ABC. Therefore the ratio of the rectangle of AB and AC to the rectangle of BC and AD is also given. But the ratio of the rectangle of AB and AC to the square of



d 8. *prop.* BC is given: Therefore *d* also the ratio of the rectangle of BC and AD to the square of BC is given; and therefore the ratio of the right line BC to the right line
e 1. 6. AD is given. (For that *e* the rectangle is to the square as AD to BC.) Now let the right line FD, given in position and magnitude, be exposed; and thereon let there be described the segment of a circle FID; capable of an angle equal to the angle A. And seeing the said angle A is given, also the angle in the segment FLD shall be
f 8. *def.* given; and therefore *f* the same segment is given in position. From the point D let there be erected at right
g 4. *def.* angles on the line FD, the line DH, which *g* is given

in



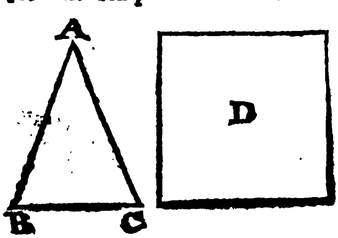
in position: Let it be so made, that as BC is to AD, so FD may be to DH; and seeing that the ratio of BC to AD is given, also that of FD to DH is given. But FD is given: Therefore *h* DH is given in magnitude. But it is also given in position, and the point D is given: Therefore the point H is *i* also given. Now through the point H let there be drawn HI, parallel to FD, and that line HI shall be given in *k* position. But the segment of the circle FID is also given in position. Therefore *l* the point I is given. Let the right lines IF and ID be drawn, and the perpendicular IE: Therefore IE is given in position. But the point I is given, as also each of the points F and D: Therefore *m* each of the lines FD, FI, and ID is given in position and magnitude: Wherefore *n* the triangle FID is given in kind; and seeing that as BC is to AE, so is FD to DH, and *o* that to DH, IE is equal; as BC is to AE, so is FD to IE, and the angle A is equal to the angle FID: Therefore *p* the triangle ABC is equiangled to the triangle FID. But FID is given in kind: Therefore also the triangle ABC is given in kind.

h 2. *prop.*
i 27. *prop.*
k 28. *prop.*
l 25. *prop.*
m 26. *prop.*
n 39. *prop.*
o 34. *prop.*
p 79. *prop.*

OTHERWISE.

Constr. Let the triangle ABC, whose angle A is given, and the ratio of the rectangle contained under AB and AC, to the square of BC, be given; I say that the triangle ABC is given in kind.

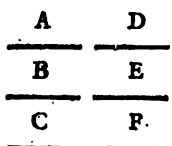
Demonstr. For seeing the angle A is given, that space by which the square of the line compounded of BAC is greater than the square of BC, *q* hath a given ratio to the triangle ABC. Now let that space be D: Therefore the ratio of D to the triangle ABC is given. But the ratio of the triangle ABC to the rectangle of AB and AC is given; *r* seeing the angle A is



q 67. *prop.*
r 66. *prop.*

s 8. prop. is given: Therefore s the ratio of the space D to the rectangle of AB and AC is given. But the ratio of the rectangle of AB and AC to the square of BC is also given: Therefore s the ratio of the space D to the square
t 6 prop. of BC is given. Wherefore by compounding, t the ratio of the space D, with the square of BC to the said square of BC is given: Therefore the ratio of the square of the line compounded of BAC, to the square of BC is given; (for that the space D with the square of BC is equal to the square of the line compounded of
u sch. 52. prop. BAC;) and therefore u the ratio of the said line compounded of BAC to BC is given. But the angle A is
x 46. prop. also given: Therefore x the triangle ABC is given in kind.

PROP. LXXXI.



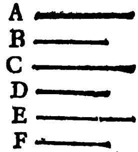
If of three right lines A, B, and C, proportional to three other proportional right lines D, E, and F, the extremes A and D, C and F, are in a given ratio (to wit, as A to D, and C to F,) also the means, B and E shall be in a given ratio, and if one extreme hath a given ratio to an extreme, and the mean to the mean, the other will have also a given ratio to the other.

Demonstr. Forasmuch as the ratio of A to D, and of
a 70. prop. C to F is given, the rectangle of A and D a shall have a given ratio to the rectangle of C and F. But the
b 17. 6. rectangle of A and D is equal b to the square of B; and the rectangle of C and F to the square of E. Therefore the ratio of the square of B to the square of E is given;
c sch. 52. prop. and therefore c the ratio of the line B to the line E is also given.

Again, let the ratio of A to D, and B to E, be given: I say that the ratio of C to F is also given. For seeing that the ratio of A to D, and of B to E is given, also
d 50. prop. the ratio of the square of B d to the square of E is given. But the square of B is equal to the rectangle of A and C, and the square of E to the rectangle of D and F: Therefore the ratio of the rectangle of A and C to the rectangle of D and F is given. But the ratio of a side
e 68. prop. A to a side D is given: Therefore e the ratio of the other side C to the other side F is also given.

PROP. LXXXII.

If there be four right lines A, B, C, and D, proportional, as the first A shall be to that line to which the second B hath a given ratio, so the third C shall be to that to which the fourth D hath a given ratio.

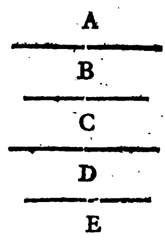


Constr. Let E be the line to which B hath a given ratio, and let it be so as that B may be to E, as D is to F.

Demonstr. Now the ratio of B to E is given, therefore also the ratio of D to F is given. And seeing that as A is to B, so is C to D. And again, as B is to E, so is D to F, by ratio of equality, as A is to E, so is C to F. But E is that line to which B hath a given ratio, and F that to which D also hath a given ratio: Therefore as A is to that to which B hath a given ratio, so C is to that to which D hath a given ratio.

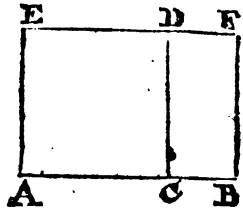
PROP. LXXXIII.

If four right lines A, B, C, and D, are in such sort to one another, that of any three of them A, B, C, and a fourth E, taken proportional, to which that line D, which remains of the four lines, hath a given ratio, it shall be as the fourth D is to the third C, so the second B shall be to that to which the first A hath a given ratio.



Demonstr. Forasmuch as A is to B, as C is to E, the rectangle contained under A and E *a* is equal to the a 16. 6.
rectangle contained under B and C; and seeing that the ratio of D to E is given, also shall be given the ratio of the rectangle of A and D to the rectangle of A and E (for *b* it is the same ratio as of D to E.) But the rectangle b 1. 6.
of A and E is equal to the rectangle of B and C. Therefore the ratio of the rectangle of A and D to the rectangle of B and C is given. Wherefore *c* as D is to C, so c 56. *prop.*
is B to that to which A hath a given ratio.

PROP. LXXXIV.

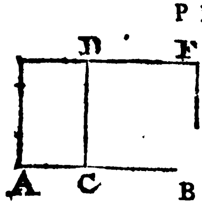


If two right lines AB and AE comprehending a given space AF in a given angle BAE, and that the one AB be greater than the other AE by a given line CB, also each of the lines AB and AE is given.

Demonstr. For seeing that AB is greater than AE by the given line CB, the remainder AC is equal to AE: Finish the parallelogram AD. Therefore seeing that AE is equal to AC, the ratio of AE to AC is given, and the angle A is also given: Therefore *a* AD is given in kind. Wherefore the given space AF is applied to the given right line CB, exceeding it by the given figure AD given in kind; and therefore *b* the breadth of the excess is given. Therefore AC is given. But CB is also given: Therefore the whole AB is given. But AE is also given: Therefore each of the right lines AB and AE is given.

a *sch.* 61. *prop.*
b 59. *prop.*

PROP. LXXXV.



If two right lines AC and CD, do comprehend a given space AD in a given angle ACD, the line compounded of those lines AC and CD is given, also each of those lines AC and CD is given.

Constr. For let AC be prolonged to the point B, and let CB be put equal to CD, then through the point B let BF be drawn parallel to CD, and so finish the parallelogram CF.

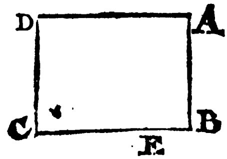
Demonstr. Seeing then that CB is equal to CD, and the angle DCB is given; for that angle that follows is the given angle; and therefore *a* the parallelogram DB is given in kind: and again, seeing that the line compounded of ACD is given, and CB is equal to CD, also AB is given. And thus to the right line AB there is applied the given space AD, deficient by the figure DB given in kind; and therefore *b* the breadths of the defects are also given: Therefore the right lines

a *sch.* 61. *prop.*
b 58. *prop.*

DC

DC and CB are given. But the compounded line ACD is alſo given: Therefore *c* each of the lines AC and CD is given. c 4. *prop.*

PROP. LXXXVI.



If two right lines AB and BC, do comprehend a given ſpace AC, in a given angle ABC, the ſquare of the one BC, is greater than the ſquare of the other AB, by a given ſpace (yet in a given ratio,) alſo each of thoſe lines AB and BC ſhall be given.

Demonſtr. For ſeeing that the ſquare of BC is greater than the ſquare of AB by a given ſpace (yet in a certain ratio:) Let the given ſpace be taken away, that is to ſay, the rectangle contained under CB and BE: Therefore *a* the ratio of the remainder, *b* which is the rectangle contained under BC and CE to the ſquare of AB is given. And foraſmuch as the rectangle † under AB and BC is given, and alſo that of CB and BE, their *c* ratio is given. But as the rectangle under AB and BC is to the rectangle under CB and EB, *d* ſo AB is to BE; and therefore the ratio of AB to BE is given: Wherefore *e* the ratio of the ſquare of AB to the ſquare of BE is alſo given. But the ratio of the ſquare of AB to the rectangle under BC and CE is given: Therefore *f* alſo the ratio of the rectangle under BC and CE to the ſquare of BE is given. Wherefore the ratio of four times the rectangle under BC and CE to the ſquare of BE is given; and by compounding, *g* the ratio of four times the rectangle under BC and CE, with the ſquare of BE to the ſquare of BE is given. But four times the rectangle of BC and CE, with the ſquare of BE, *h* is the ſquare of the compound line BCE: Therefore the ratio of the ſquare of the compound line BCE to the ſquare of BE is given: Wherefore *i* the ratio of the line compounded of BC and CE to BE is given, and by compounding, *k* the ratio of the compound of the lines BC, CE, and BE, that is to ſay, the double of BC to BE is given; and therefore the ratio of the only line BC to BE is alſo given. But as BC is to BE, *l* ſo the rectangle under BC and BE is to the ſquare of BE: Therefore the ratio of the rectangle under BC and BE to the ſquare of BE is given. But

a 11. *def.*
b 2. 2.
c 1. *prop.*
d 1. 6.
e 50. *prop.*
f 8. *prop.*
g 6. *prop.*
h 8. 2.
i 54. *prop.*
k 6. *prop.*
l 1. 6.

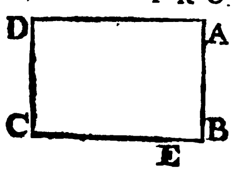
m 2. prop. But the rectangle of BC and BE is given: Therefore *m* the square of BE is also given, and consequently the line BE is given. Wherefore BC is also given, seeing that the ratio of BE to BC is given. But the space
n 57. prop. AC is given, and also the angle B: Therefore *n* AB is given. Wherefore each of the lines AB and BC is given.

Scholium.

† *Instead of saying in this place* [what is under, &c.] *we have used this word rectangle, it being manifest by what follows that such was the intention of* EUCLIDE, *seeing he makes use in the said Demonstration of the second and eighth proposition of the second Element; and also that the space or parallelogram given being not rectangled, it may be reduced thereto, making on BC, and in the given point B, a right angle CBA, so as that there will be two parallelograms constituted on one and the same base BC, and between the same parallels, as in the 69th proposition, by means whereof this conclusion is drawn.*

Note, This serves also for the next Prop.

PROP. LXXXVII.



If two right lines AB and BC, do comprehend a given space AC, in a given angle B, the square of the one BC is greater than the square of the other AB, by a given space; also each of those lines AB and BC shall be given.

Demonstr. For seeing that the square of BC is greater than the square of AB by a given space: Let the given space be taken away, and let the rectangle be contained under BC and BE: There-
a 2. 2. fore the remainder *a* which is the rectangle of BC and CE, is equal to the square of AB. And seeing that the rectangle of BC and BE is given, and also the space or rectangle AC, the ratio of the said rectangle of BC and
b 1. 6. BE to AC is given. But as *b* the rectangle of BC and BE is to the rectangle of AB and BC, so is BE to AB: Therefore the ratio of BE to AB is given, and therefore
c 50. prop. *c* the ratio of the square of the said DE to the square of AB is also given. But to that square of AB the rectangle of BC and CE is equal: Therefore the ratio of the said rectangle of BC and CE to the square of BE

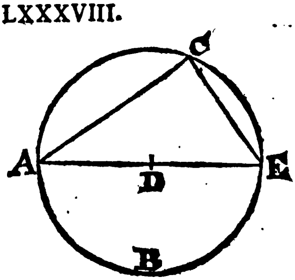
is

is given; and therefore the ratio of the quadruple of the said rectangle of BC and CE to the square of BE is also given; and by compounding, *d* the ratio of four times the rectangle of BC and CE, with the square of BE to the said square of BE is given. But four times the rectangle of BC and CE, with the square of BE; *e* is the square of the compound line BCE: Therefore the ratio of the square of that compound line BCE to the square of BE is also given; and therefore the ratio *f* of the compound line BCE to BE is given. Wherefore by compounding *g* the ratio of the said compound line BCE and EB, that is to say, twice BC to BE is also given; therefore the ratio of the only line BC to BE is given. But the ratio of the same BE to AB is also given: Therefore *h* the ratio of AB to BC is given. And seeing that the ratio of BC to BE is given, and that as the said BC is to BE, so the square of BC *i* to the rectangle of BC and BE, the ratio of the square of BC to the rectangle of BC and BE is also given. But the said rectangle of BC and BE is given, it being that which was taken away, and which was given. Therefore the square of BC *k* is given, and therefore the line BC is given. But the ratio of the same BC to BA is given, therefore AB is also given.

d 6. *prop.*
e 8. 2.
f 54. *prop.*
g 6. *prop.*
h 8. *prop.*
i 1. 6.
k 2. *prop.*

PROP. LXXXVIII.

If in a circle ABC, given in magnitude, there be drawn a right line AC, which shall take away a segment ABC, which doth comprehend a given angle AEC, that line AC is given in magnitude.



Constr. For let D be the center of the circle; and let the diameter thereof ADE be drawn, and let EC be joined.

Demonstr. The angle ACE is given, for *a* it is a right angle. But the angle AEC is also given, and therefore the other angle CAE is given. Wherefore the triangle ACE *b* is given in kind; and therefore the ratio of EA to AC is given. But AE is given in magnitude, seeing that the circle ABC is given in magnitude. Therefore *c* AC is also given in magnitude.

a 31. 3.
b 40. *prop.*
c 2. *prop.*

PROP. LXXXIX.

If in a circle ABC, given in magnitude, there be drawn a right line AC, given in magnitude, that line AC will take away a segment ABC, comprehending a given angle.

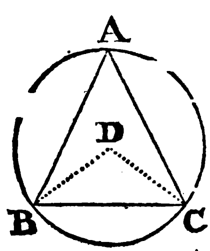
Constr. For having taken the point D for the center of the circle, let the diameter ADE be drawn, as also the right line EC.

Demonstr. Forasmuch as each of the right lines AE
a 1. *prop.* and AC are given, the ratio of the line AE to AC *a* is given; and the angle ACE is a right angle: Therefore
b 43. *prop.* *b* the triangle ACE is given in kind, and therefore the angle AEC is given.

PROP. XC.

If in the circumference of a circle ABC, given in position and in magnitude, there be taken a given point B, and that from the point B to the circumference of the circle ABC, a right line BAC be inflected so as to make a given angle BAC, the other extremity C of the inflected line shall be given.

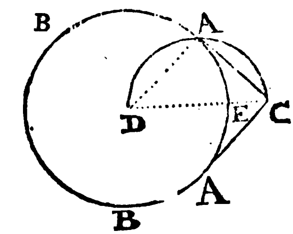
Constr. For let the center of the circle be D, and let the right lines BD and BC be drawn.



Demonstr. Forasmuch as each
a 26. *prop.* point B and D is given, the right line BD, *a* is given in position; and seeing that the angle BAC is given, the angle BDC is also given. Wherefore to the right line BD, given in position, and in the point D given therein, there is drawn the right line CD; which makes the given
b 29. *prop.* angle BDC; and therefore *b* the line DC is given in position. But the circle ABC is given in position and magnitude:
c 6. *def.* Therefore *c* the right line DC is given in position and
d 27. *prop.* in magnitude. But the point D is given: Therefore *d* the point C is also given.

PROP.

PROP. XCI.



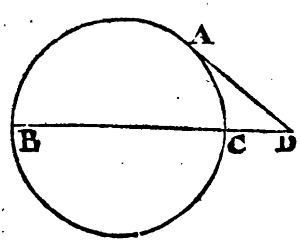
If from a given point C, there be drawn a right line CA, which shall touch a circle AB, given in position; that line CA is given in position and in magnitude.

Constr. For having taken the point D for the center of the circle, let the right lines DA and DC be drawn.

Demonstr. Forasmuch as each point C and D is given, the right line CD *a* is given in position and in magnitude. But the angle CAD *b* is a right angle; and therefore the semicircle described on CD shall pass through the point A: Let it then pass through that point, and let the semicircle be DAC: Forasmuch as the same DAC *c* is given in position, and also the circle ABE, *d* the point A is given. But the point C is also given: Therefore *e* the right line AC is given in position and in magnitude.

a 26. *prop.*
b 18. 3.
c 6. *def.*
d 25. *prop.*
e 26. *prop.*

PROP. XCII.



If without a circle ABC, given in position, there be taken some point D, and from that given point there be drawn a right line DB, cutting the circle, the rectangle comprised under the whole line BD, and the part DC, between the point D, and the convexity of the circumference AC, shall be given.

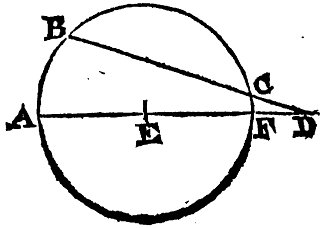
Constr. For from the point D let the right line DA be drawn, which shall touch the circle in the point A.

Demonstr. Therefore DA *a* is given in position and magnitude; and therefore the square of the said DA is *b* given. But the said square of DA is equal *c* to the rectangle of BD and DC: Therefore the said rectangle of BD and DC is also given.

a 91. *prop.*
b 52. *prop.*
c 36. 3.

OTHER-

OTHERWISE.

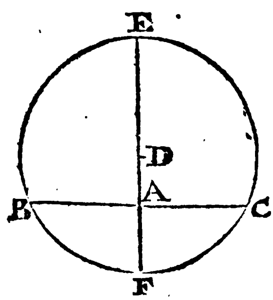


Constr. Let E be the center of the circle, and through the same center let there be drawn from the point D the right line DA.

Demonstr. Forasmuch as each point D and E is given, the right line DE is *d* given in position and in magnitude. But the circle ABC is given in position and in magnitude: Therefore each point A and F *e* is given, and the point D is also given; and therefore each line AD and FD is given. Wherefore the rectangle of the lines AD and DF is also given. But the said rectangle of AD and DF is equal to the rectangle of DB and DC: Therefore the rectangle of DB and DC is given.

d 26. *prop.*
e 25. *prop.*

PROP. XCIII.



If in a circle given in position there be taken a given point A, and through that point A there be drawn a right line BC to the circle, the rectangle comprised under the segments of the same line BC shall be given.

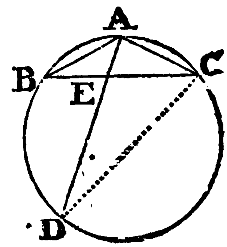
Constr. For let D be taken for the center of the circle, and having drawn the right line AD prolong it to the points E and F.

Demonstr. Forasmuch as each point A and D is given, the right line AD *a* is given in position. But the circle BEC is also given in position: Therefore each point E and F is also given in position, and the point A is given. Wherefore each line *b* AE and AF is given: Therefore the rectangle of the same lines AE and AF is given, and is equal to the rectangle *b* of AB and AC: Therefore the said rectangle of AB and AC is given.

a 26. *prop.*
b 35. 3.

PROP.

PROP. XCIV.



If in a circle ABC, given in magnitude, there be drawn a right line BC, which doth take away a segment which doth comprehend a given angle BAC, and that the said angle in the segment is cut into two equal parts, the line compounded of the right lines BA and AC, which comprehend the given angle BAC shall have a given ratio to the line AD, which doth divide that angle into two equal parts; and the rectangle contained under the line compounded of those lines BA and AC, comprehending the given angle BAC, and that part ED of the intersecting line which is below the segment between the base BC and the circumference, shall be given.

Constr. Let BD be drawn.

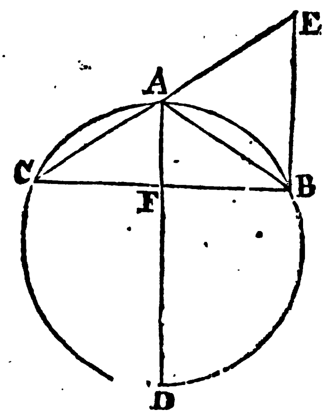
Demonstr. Forasmuch as in the circle ABC given in magnitude, there is drawn the right line BC, which takes away the segment BAC, comprehending the given angle BAC, that line BC *a* is given; and therefore BD is also given: Therefore the ratio of BC to BD *b* is given. And seeing that the given angle BAC is cut in two equal parts by the right line AD, as *c* BA is to CA, so is BE to CE; and by compounding, as BAC is to CA, so is BC to CE; and by permutation, as BAC is to BC, so is CA to CE. And seeing that the angle BAE is equal to the angle CAE, and the angle ACE *d* to the angle BDE, the other angle AEC is equal to the other angle ABD; and therefore the triangle ACE is equiangled to the triangle ABD: Therefore *e* as AC is to CE, so is AD to BD. But as CA is to CE, so the line compounded of BA and AC is to BC: Therefore as the compound line BAC is to BC, so is AD to BD; and by permutation, as the compound line BAC is to AD, so is BC to BD. But the ratio of BC to BD is given: Therefore the ratio of the compound line BAC to AD is also given. Moreover, I say that the rectangle under the compound line BAC and ED is given. For seeing that the triangle AEC is equiangled to the triangle BDE, (for the angle ACE *d* is equal to the angle BDE, and the angle AEC

a 88. *prop.*
b 1. *prop.*
c 3. 6.
d 21. 3.
e 4. 6.

f 15. 1. f to the angle BED) as BD is to DE, so is AC to CE. But as AC is to CE, so is also the compound line BAC to BC: Therefore as the compound line BAC is to BC, so is BD to DE. Wherefore the rectangle of the compound line BAC and DE g is equal to the rectangle of BC and BD. But the rectangle of BC and BD is given, (for that those lines BC and BD are given:) Therefore the rectangle under the compound line BAC and ED is also given.

g 16. 6.

OTHERWISE.



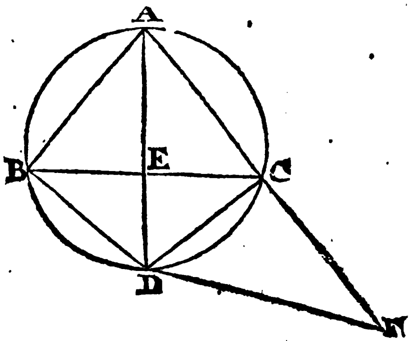
Constr. Let CA be prolonged to the point E, and let AE be put equal to BA, and let BE and BD be joined.

Demonstr. Forasmuch as the angle BAC is double to each of the angles CAD and AEB (for the angle BAC is cut into two equal parts by the line AD, and equal *h* to the two angles ABE and AEB, which *i* are equal) the angle ABE is equal to the angle CAD, that is to say, *k* to the angle CBD; adding therefore the common angle ABC, the whole angle ABD shall be equal to the whole angle FBE. But the angle ACB is *k* equal to the angle ADB: Therefore the third angle AEB is equal to the third angle BAD; and therefore the triangle CEB is equiangled to the triangle ABD: Wherefore as CE is to CB, so is AD to BD. But the right line CE is compounded of the two lines CA and AB: Therefore as the compound line BAC is to CB, so is AD to BD; and by permutation, as the compound line BAC is to AD, so is CB to BD. But the ratio of CB to BD is given, seeing that each of those lines is given: Therefore the ratio of the compound line BAC to AD is also given. And seeing that the triangle CEB is equiangled to the triangle FBD (for the angle AFC is equal *l* to the angle BFD, and the angle ECB *m* to the

h 32. 1.
i 5. 1.
k 21. 3.
l 21. 3.
m 16. 6.

the angle ADB) as EC is to CB, so is BD to DF. But EC is equal to the compound line BAC: Therefore as the compound line BAC is to CB, so is BD to DF. Wherefore *n* the rectangle of the compound line BAC and DF is equal to the rectangle of CB and BD. But the rectangle of CB and BD is given, since each of the lines CB and BD is given: Therefore the rectangle of the compound line BAC and DF is given. n 16. 6.

OTHERWISE.

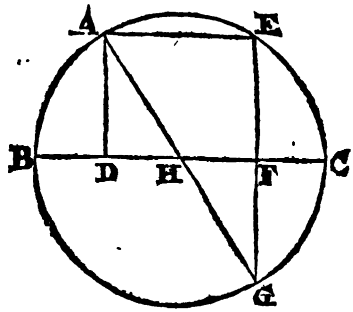


Constr. Let AC be prolonged to F, and let CF be put equal to AB, and let the right lines BD and DF be drawn.

Demonstr. Forasmuch as BA is equal to CF, and *o* BD to DC, the two sides AB and BD are equal to the two sides CD and DF, each to his corresponding side, and the angle ABD is equal to the angle DCF, *p* seeing that the four sided figure ABDC is within the circle: Therefore the base AD is *q* equal to the base DF, and the angle DAB to the angle DFC. But the angle BAD is given, being the half of the given angle BAC, therefore the angle DFC is so also. But DAF is also given: Therefore the triangle ADF is given in kind. Wherefore the ratio of FA to AD is given. But AF is the compound of BA and AC, for that CF is equal to AB: Therefore the ratio of the compound line BAC to AD is given: The same demonstration will serve to shew that the rectangle contained under the compound line BAC and ED is given also. o 26, 29. 3. p 22. 3. q 4. 1.

PROP.

PROP. XCV.



If in the diameter BC of a circle ABC given in position, there be taken a given point D, and from that point D there be drawn a right line DA, to the circumference of the circle. And if from the section of the said line there be drawn a right line AE, perpendicular thereto, and through the point E where that perpendicular doth meet with the circumference, there be drawn a parallel EF, to the first line drawn AD, that point F in which the parallel meets with the diameter, is given; and the rectangle contained under the parallel lines AD and EF is also given.

Constr. Let the right line EF be prolonged to the point G, and let the right line AG be drawn.

Demonstr. Forasmuch as the angle AEG is a right angle, the right line AG is the diameter of the circle. But BC is also the diameter: Therefore the point H is the center of the circle. Now the point D is given;
a 26. *prop.* and therefore *a* the line DH is given in magnitude. But seeing that AD is parallel to EG, and AH equal to GH;
b 26. *prop.* *b* DH is equal to FH, and AD to FG: (for the angles
c 15. 1. AHD and FHG *c* are equal, and DAH and FGH *d* are
d 26. 1. also equal.) But the line DH is given: Therefore FH is also given. But each of those lines DH and HF is also given in position, and the point H is given: Therefore
e 27. *prop.* *e* the point F is also given. And seeing that in the circle ABC given in position, is taken the given point F, and through the same is drawn the right line
f 93. *prop.* EFG; the rectangle under EF and FG *f* is given. But FG is equal to AD. Therefore the rectangle comprehended under AD and EF is given. *Which was to be demonstrated.*

The End of EUCLIDE's *DATA.*

A BRIEF TREATISE (Added by *F. LUSSAS*) OF Regular Solids.

REgular Solids are ſaid to be compoſed and mix'd, when each of them is transformed into other Solids, keeping ſtill the form, number and inclination of the baſes, which they before had to one another; ſome of which yet are transformed into mix'd Solids, and other ſome into ſimple. Into mix'd, as a Dodecaedron and an Icoſaedron, which are transformed or altered, if you divide their ſides into two equal parts, and take away the ſolid angles ſubtended by plane ſuperficial figures, made by the lines coupling thoſe middle ſections; for the Solid remaining after the taking away of thoſe ſolid angles, is called an Icoſidodecaedron. If you divide the ſides of a Cube and of an

Octoedron

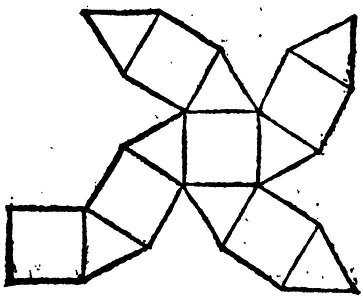
Octoedron into two equal parts, and couple the sections, the solid angles subtended by the plane superficies made by the coupling lines, being taken away, there shall be left a Solid, which is called an Exoctoedron. So that both of a Dodecaedron, and also of an Icosaedron, the Solid which is made shall be called an Icosidodecaedron; and likewise the Solid made of a Cube, and also of an Octoedron, shall be called an Exoctoedron. But the other Solid, to wit, a Pyramis or Tetraedron, is transformed into a simple Solid; for if you divide into two equal parts each of the sides of the Pyramis, triangles described of the lines which couple the sections, and subtending and taking away the solid angles of the Pyramis, are equal and like unto the equilateral triangles left in each of the bases, of all which triangles is produced an Octoedron, to wit, a simple, and not a composed Solid. For the Octoedron hath four bases, like in number, form, and mutual inclination with the bases of the Pyramis, and hath the other four bases with like situation opposite and parallel to the former. Wherefore the application of the Pyramis taken twice, maketh a simple Octoedron, as the other Solids make a mix'd compound Solid.

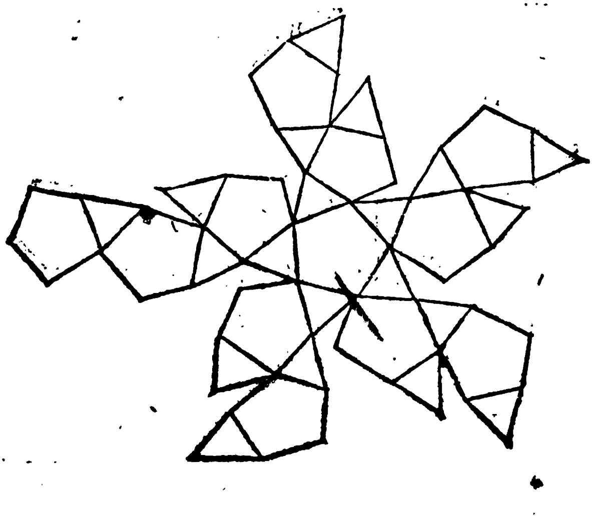
DEFI-

DEFINITIONS.

I.

An Exoctoedron is a solid figure contained of six equal squares, and eight equilateral and equal triangles.





II.

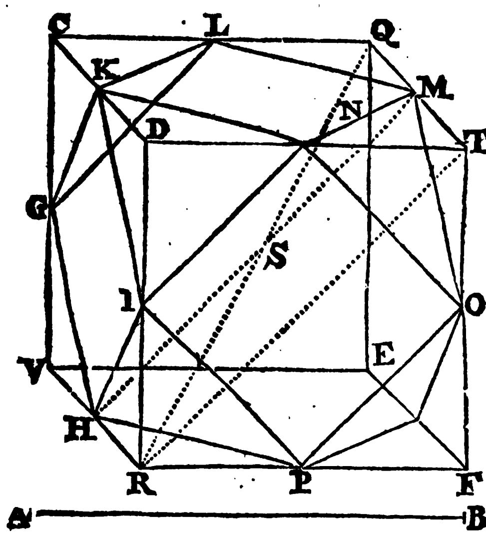
An Icosidodecaedron is a solid figure contained under twelve equilateral, equal, and equiangled Pentagons, and twenty equal and equilateral triangles.

For the better understanding of the two former Definitions, and also of the two Propositions following, I have here set two figures, which if you first describe upon paste-board, or such like matter, and then cut them and fold them accordingly, they will represent unto you the perfect forms of an Exoctoedron, and of an Icosidodecaedron.

PROBLEM I.

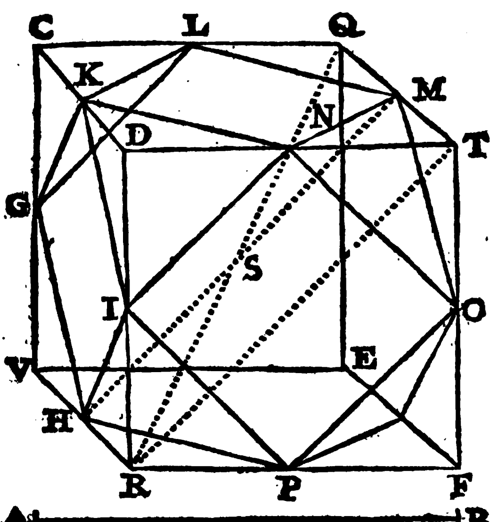
To describe an equilateral and equiangled Exoctoedron, and to contain it in a given sphere, and to prove that the diameter of the sphere is double to the side of the said Exoctoedron.

Constr. Suppose a Sphere whose diameter let be AB, and about the diameter AB let there be described a square



a 6. 4. *a*, and upon the square let there be described a Cube *b*,
b 15. 13. which let be CDEFQTVR; and let the diameter thereof be

be QR, and the center S. Divide the sides of the Cube into two equal parts in the points G, H, I, K, L, M, N, O, P, &c. and couple the middle sections by the right lines IN, NO, OP, PI, and such like, which subtend the angles of the squares or bases of the Cube; and they are equal c, and contain right angles, as the angle NIP. c 4. 1.
For the angle NID, which is at the base of the Isosceles triangle NDI, is the half of a right angle, and so likewise is the opposite angle RIP. Wherefore the residue NIP is a right angle, and so the rest. Wherefore NIPO is a square. And for the same reason shall the rest NMLK, KGHI, &c. inscribed in the bases of the Cube, be squares, and they shall be six in number, according to the number of the bases of the Cube. Again, forasmuch as the triangle RIN subtendeth the solid angle D of the Cube, and likewise the triangle KGL the solid angle C, and so the rest which subtend the right solid angles of the Cube, and these triangles are equal and equilateral (to wit) being made of equal sides, and they are the li-



mits or borders of the squares, and the squares the limits or borders of them; as hath been before proved. Where-

 fore

fore LMNOPHGK is an Exoctoedron by the definition, and is equilateral, for it is contained by equal subtendant lines; it is also equiangled, for every solid angle thereof is contained under two superficial angles of two squares, and two superficial angles of two equilateral triangles.

Demonstr. Forasmuch as the opposite sides and diameters of the bases of the Cube are parallels, the plane extended by the right lines QT and VR, shall be a parallelogram, And for that also in that plane lieth QR, the diameter of the Cube, and in the same plane also is the line MH, which divideth the said plane into two equal parts, and also coupleth the opposite angles of the Exoctoedron: this line MH therefore divideth the diame-
d *cor*. 34. 1. ter into two equal parts *d*; and also divideth it self
e 4. 1. in the same point, which let be S, into two equal parts *e*. And by the same reason may we prove that the rest of the lines which couple the opposite angles of the Exoctoedron, do in S the center of the Cube, divide one another into two equal parts, for each of the angles of the Exoctoedron are set in each of the bases of the Cube. Wherefore making the center the point S, with the distance SH or SM describe a Sphere, and it shall touch every one of the angles equidistant from the point S.

And forasmuch as AB the diameter of the sphere given, is put equal to the diameter of the base of the cube, to wit, to the line RT, and the same line RT is equal
f 33. 1. to the line MH *f*, which line MH coupling the opposite angles of the Exoctoedron, is drawn by the center. Wherefore it is the diameter of the Sphere given which containeth the Exoctoedron.

Lastly, forasmuch as in the triangle RFT, the line PO doth cut the sides into two equal parts, it shall cut them proportionally with the bases, to wit, as FR is
g 2. 6. to FP, so shall RT be to FO *g*. But FR is double to FP by supposition: Wherefore RT, or the diameter HM, is also double to the line PO, the side of the Exoctoedron. Wherefore we have described, &c. Which was required to be done.

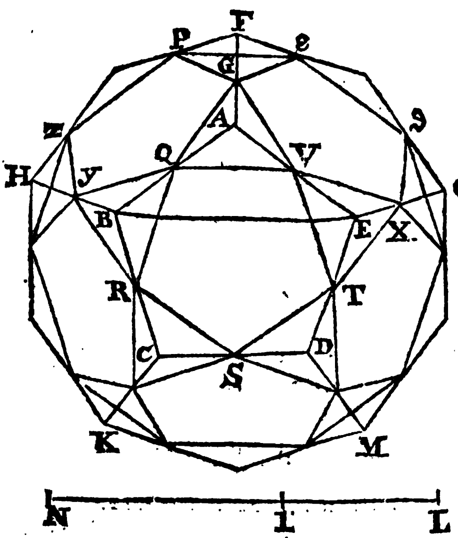
PROBLEM II.

To describe an equilateral and equiangled Icosidodecaedron, and to comprehend it in a sphere given, and to prove that the diameter being divided in extream and mean proportion, maketh the greater segment double to the side of the Icosidodecaedron.

Constr.

Constr. Suppose that the diameter of the sphere given be NL, *a* divide the line NL, in extream and mean proportion in the point I, and the greater segment thereof let be NI, and upon the line NI describe a Cube *b*; and about this Cube let there be circumscribed a Dodecaedron *c*; and let the same be ABCDEFHKMO, and divide each of the sides into two equal parts in the points Q, R, S, T, V, X, Y, Z, P, ε, ϑ, G, &c. and couple the sections with right lines, which shall subtend the angles of the Pentagons, as the lines PG, GV, VQ, QY, YR, RQ, VT, TX, XV, and so the rest. a 30. 6. b. 15. 13. c. 17. 13.

Demonstr. Forasmuch as these lines subtend equal angles of the Pentagons, and those equal angles are contained by equal sides, to wit, by the half of the sides of the Pentagons; therefore those subtending lines are equal *d*. Wherefore the triangles GQV, YQR, and VXT, and the rest, which take away solid angles of the Dodecaedron, are equilateral. d. 4. 1.



Again, forasmuch as in every Pentagon are described five equal right lines, coupling the middle sections of the sides, as are the lines QV, VT, TS, SR, and RQ, they describe a Pentagon in the plane of the Pentagon of

 the

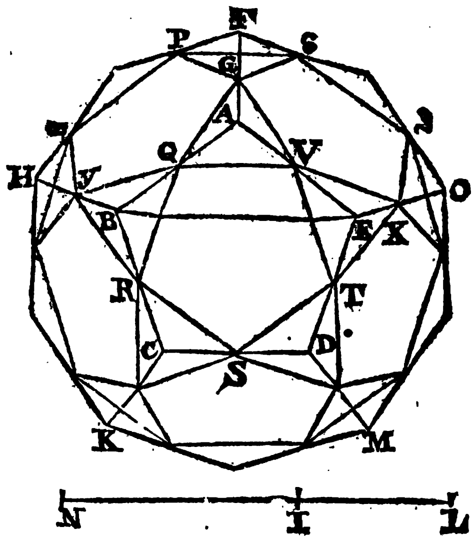
the Dodecaedron. And the said Pentagon is contained in a circle, to wit, whose center is the center of a Pentagon of the Dodecaedron. For the lines drawn from that center to the angles of this Pentagon are equal, for that
e 12. 4. they are perpendiculars upon the bases cut *e*. Wherefore
f 11. 4. the Pentagon QRSTV, is equiangled *f*. And by the same reason may the rest of the Pentagons described in the bases of the Dodecaedron, be proved equal and like.

Wherefore those Pentagons are twelve in number: And forasmuch as the equal and like triangles do subtend and take away twenty solid angles of the Dodecaedron; therefore the said triangles shall be twenty in number. Wherefore we have described an Icosidodecaedron by the definition, which Icosidodecaedron is equilateral; for that all the sides of the triangles are equal and common with the Pentagons; and it is also equiangled. For each of the solid angles is made of two superficial angles of an equilateral Pentagon, and of two superficial angles of an equilateral triangle.

Now let us prove that it is contained in the given sphere whose diameter is NL. Forasmuch as perpendiculars drawn from the centers of the Dodecaedron, to the middle sections of his sides, are the halfs of the lines which couple the opposite middle sections of the sides of
g 3. cor. of 17. 13. the Dodecaedron *g*; which lines also *h* do in the center
h idem. divide one another into two equal parts. Therefore right lines drawn from that point to the angles of the Icosidodecaedron (which are set in those middle sections) are equal; which lines are thirty in number, according to the number of the sides of the Dodecaedron, for each of the angles of the Icosidodecaedron are set in the middle sections of each of the sides of the Dodecaedron. Wherefore making the center of the Dodecaedron, and the space, any one of the lines drawn from the center to the middle sections, describe a sphere, and it shall pass through all the angles of the Icosidodecaedron, and shall contain it.

And forasmuch as the diameter of this solid, is that right line whose greater segment is the side of the Cube
i 4. cor. 17. 13. inscribed in the Dodecaedron *i*, which side is NI by supposition. Wherefore that solid is contained in the sphere given, whose diameter is put to be the line NL.

Now



Now let us prove that the great segment of the diameter is double to QV the side of the solid. Forasmuch as the sides of the triangle AEB are in the points Q and V divided into two equal parts, the lines QV and BE are parallels *k*. Wherefore as AE is to AV, so is EB to VQ *l*. But the line AE is double to the line AV. Wherefore the line BE is double to the line QV *m*. Now the line BE is equal to NI, or to the side of the Cube *n*; which line NI is the greater segment of the diameter NL. Wherefore the greater segment of the diameter given is double to the side of the Icosidodecaedron inscribed in the given sphere. Wherefore we have described, &c. Which was required to be done.

k *cor.* 39. 1.
l 2. 6.
m 4. 6.
n 2 *cor.* of 17. 13.

ADVERTISEMENT.

To the understanding of the nature of this Icosidodecaedron, you must well conceive the passions and proprieties of both these solids, of whose bases it consisteth, to wit, of the Icosaedron and of the Dodecaedron. And altho' in it the bases are placed oppositely, yet have they to one another one and the same inclination. By reason whereof there lie hidden in it the actions and passions

of the other regular Solids. And I would have thought it not impertinent to the purpose to have set forth the inscriptions and circumscriptious of this Solid, if want of time had not hindred. But to the end the Reader may the better attain to the understanding thereof, I have here following briefly set forth, how it may in or about every one of the five regular Solids be inscribed or circumscribed; by the help whereof he may, with small travel, or rather none at all, having well poised and considered the Demonstrations appertaining to the foresaid five regular Solids, demonstrate both the inscription of the said Solids in it, and the Inscription of it in the said Solids.

Of the Inscriptions and Circumscriptions of an Icosidodecaedron.

An Icosidodecaedron may contain the other five regular bodies. For it will receive the angles of a Dodecaedron in the centers of the triangles which subtend the solid angles of the Dodecaedron, which solid angles are twenty in number, and are placed in the same order in which the solid angles of the Dodecaedron, taken away, or subtended by them, are. And for that reason it shall receive a Cube and a Pyramis contained in the Dodecaedron, when as the angles of the one are set in the angles of the other.

An Icosidodecaedron receiveth an Octoedron, in the angles cutting the six opposite sections of the Dodecaedron, even as if it were a simple Dodecaedron.

And it containeth an Icosaedron, placing the twelve angles of the Icosaedron in the same centers of the twelve Pentagons.

It may also by the same reason be inscribed in each of the five regular bodies, to wit, in a Pyramis, if you place four triangular bases concentrical with four bases of the Pyramis, after the same manner that you inscribed an Icosaedron in a Pyramis; so likewise may it be inscribed in an Octoedron, if you make eight bases thereof concentrical with the eight bases of the Octoedron. It shall also be inscribed in a Cube, if you place the angles which receive the Octoedron in it, in the centers of the bases of the Cube. Again, you shall inscribe it in an Icosaedron, when the triangles compassed in of the Pentagon bases, are concentrical with the triangles which make a solid angle of the Icosaedron.

Lastly,

Lastly, it shall be inscribed in a Dodecaedron, if you place each of the angles thereof in the middle sections of the sides of the Dodecaedron, according to the order of the construction thereof.

The opposite plain superficies also of this solid are parallels. For the opposite solid angles are subtended of parallel plain superficies, as well in the angles of the Dodecaedron subtended by triangles, as in the angles of the Icosaedron subtended of Pentagons, which thing may easily be demonstrated. Moreover, in this solid are infinite properties and passions, springing from the solids whereof it is composed.

Wherefore it is manifest, that a Dodecaedron and an Icosaedron mixed, are transformed into one and the self same solid of an Icosidodecaedron. A Cube also and an Octoedron are mixed and altered into another solid, to wit, into one and the same Exoctoedron. But a Pyramis is transformed into a simple and perfect solid, to wit, into an Octoedron.

If we will frame these two solids joined together into one solid, this only must we observe.

In the Pentagon of a Dodecaedron inscribe a like Pentagon, and let its angle be set in the middle sections of the Pentagon circumscribed, and then upon the said Pentagon inscribed, let there be set a solid angle of an Icosaedron, and so observe the same order in each of the bases of the Dodecaedron, and the solid angles of the Icosaedron, set upon these Pentagons, shall produce a solid consisting of the whole Dodecaedron, and whole Icosaedron. In like sort, if in every base of the Icosaedron, the sides being divided into two equal parts, be inscribed an equilateral triangle, and upon each of those equilateral triangles be set a solid angle of a Dodecaedron, there shall be produced the same solid consisting of the whole Icosaedron, and of the whole Dodecaedron.

And after the same order, if in the bases of a Cube be inscribed squares subtending the solid angles of an Octoedron, or in the bases of an Octoedron be inscribed equilateral triangles subtending the solid angles of a Cube, there shall be produced a solid consisting of either of the whole solids, to wit, of the whole Cube, and of the whole Octoedron.

But equilateral triangles inscribed in the bases of a Pyramis, having their angles set in the middle sections of the sides of the Pyramis, and the solid angles of a Pyramis,

ramis, set upon the said equilateral triangles, there shall be produced a solid consisting of two equal and like pyramids.

And now if in these solids thus composed, you take away the solid angle, there shall be restored again the first composed solids, to wit, the solid angles taken away from a Dodecaedron and an Icosaedron composed into one, there shall be left an Icosidodecaedron, the solid angles taken away from a Cube and an Octoedron composed into one solid, there shall be left an Exoctoedron. Moreover, the solid angles taken away from two pyramids composed into one solid, there shall be left an Octoedron.

Of the nature of a trilateral and equilateral Pyramis.

1. A trilateral equilateral Pyramis is divided into two equal parts, by three equal squares, which in the center of the Pyramis cut one another into two equal parts, and perpendicularly, and whose angles are set in the middle sections of the sides of the Pyramis.

2. From a Pyramis are taken away four Pyramids like unto the whole, which utterly take away the sides of the Pyramis, and that which is left is an Octoedron, inscribed in the Pyramis, in which all the solids inscribed in the Pyramis are contained.

3. A perpendicular drawn from the angle of the Pyramis to the base, is double to the diameter of the Cube inscribed in it.

4. And a right line coupling the middle sections of the opposite sides of the Pyramis is triple to the side of the same Cube.

5. The side also of a Pyramis is triple to the diameter of the base of the Cube.

6. Wherefore the same side of the Pyramis is in power double to the right line which coupleth the middle section of the opposite sides.

7. And it is in power sesquialter to the perpendicular which is drawn from the angle to the base.

8. Wherefore the perpendicular is in power sesquitertia to the line which coupleth the middle sections of the opposite sides.

9. A Pyramis and an Octoedron inscribed in it, also an Icosaedron inscribed in the same Octoedron, do contain one and the same sphere.

Of the nature of an Octoedron.

1. Four perpendiculars of an Octoedron, drawn in four bases thereof from two opposite angles of the said Octoedron, and coupled together by those four bases, describe a Rhombus, or Diamond figure; one of whose diameters is in power double to the other diameter.

2. For it hath the same proportion that the diameter of the Octoedron hath to the side of the Octoedron.

3. An Octoedron and an Icosaedron inscribed in it, do contain one and the same sphere.

4. The diameter of the solid of the Octoedron is in power sesquialter to the diameter of the circle which containeth the base, and is in power duple superbipartiens tertias (that is, as 8 to 3,) to the perpendicular or side of the foresaid Rhombus; and moreover is in length triple to the line which coupleth the centers of the next bases.

5. The angle of the inclination of the bases of the Octoedron, doth, with the angle of the inclination of the bases of the Pyramis, make angles equal to two right angles.

Of the nature of a Cube.

1. The diameter of a Cube is in power sesquialter to the diameter of his base.

2. And is in power triple to his side.

3. And unto the line which coupleth the centers of the next bases, it is in power sextuple.

4. Again, the side of the Cube, is to the side of the Icosaedron inscribed in it, as the whole is to the greater segment.

5. Unto the side of the Dodecaedron, it is as the whole is to the lesser segment.

6. Unto the side of the Octoedron it is in power duple.

7. Unto the side of the Pyramis it is in power subduple.

8. Again, the Cube is triple to the Pyramis, but to the Cube the Dodecaedron is in a manner double. Wherefore the same Dodecaedron is in a manner sextuple to the said Pyramis.

Of

Of the nature of the Icosaedron.

1. Five triangles of an Icosaedron, do make a solid angle, the bases of which triangles make a Pentagon. If therefore from the opposite bases of the Icosaedron be taken the other Pentagon by them described, these Pentagons shall in such sort cut the diameter of the Icosaedron which coupleth the foresaid opposite angles, that that part which is contained between the planes of these two Pentagons shall be the greater segment, and the residue which is drawn from the plane to the angle, shall be the lesser segment.

2. If the opposite angles of two bases joined together, be coupled by a right line, the greater segment of that right line is the side of the Icosaedron.

3. A line drawn from the center of the Icosaedron to the angles, is in power quintuple to half that line which is taken between the Pentagons, or of the half of that line which is drawn from the center of the circle which containeth the foresaid Pentagon, which two lines are therefore equal.

4. The side of the Icosaedron containeth in power either of them, and also the lesser segment, to wit, the line which falleth from the solid angle to the Pentagon.

5. The diameter of the Icosaedron containeth in power the whole line, which coupleth the opposite angles of the bases joined together, and the greater segment thereof, to wit, the side of the Icosaedron.

6. The diameter also is in power quintuple to the line which was taken between the Pentagons, or to the line which is drawn from the center to the circumference of the circle which containeth the Pentagon composed of the sides of the Icosaedron.

7. The dimetient containeth in power the right line which coupleth the centers of the opposite bases of the Icosaedron, and the diameter of the circle which containeth the base.

8. Again, the said dimetient containeth in power the diameter of the circle which containeth the Pentagon, and also the line which is drawn from the center of the same circle to the circumference: that is, it is quintuple to the line drawn from the center to the circumference.

9. The

9. The line which coupleth the centers of the opposite bases containeth in power the line which coupleth the centers of the next bases, and also the rest of that line of which the side of the Cube inscribed in the Icosaedron is the greater segment.

10. The line which coupleth the middle sections of the opposite sides, is triple to the side of the Dodecaedron inscribed in it.

11. Wherefore, if the side of the Icosaedron and the greater segment thereof be made one line, the third part of the whole is the side of the Dodecaedron inscribed in the Icosaedron.

Of the Dodecaedron.

1. The diameter of a Dodecaedron containeth in power the side of the Dodecaedron, and also that right line to which the side of the Dodecaedron is the lesser segment, and the side of the Cube inscribed in it is the greater segment, which line is that which subtendeth the angle of the inclination of the bases, contained under two perpendiculars of the bases of the Dodecaedron.

2. If there be taken two bases of the Dodecaedron, distant from one another by the length of one of the sides, a right line coupling their centers being divided in extreme and mean proportion, maketh the greater segment the right line which coupleth the centers of the next bases.

3. If by the centers of five bases set upon one base, be drawn a plain superficies, and by the centers of the bases which are set upon the opposite base, be drawn also a plain superficies, and then be drawn a right line, coupling the centers of the opposite bases, that right line is so cut, that each of his parts set without the plain superficies, is the greater segment of that part which is contained between the planes.

4. The side of the Dodecaedron is the greater segment of the line which subtendeth the angle of the Pentagon.

5. A perpendicular line drawn from the center of the Dodecaedron to one of the bases, is in power quintuple to half the line which is between the planes.

6. And therefore the whole line which coupleth the centers of the opposite bases, is in power quintuple to the whole line which is between the said planes.

7. The

7. The line which subtendeth the angle of the base of the Dodecaedron, together with the side of the base, are in power quintuple to the line which is drawn from the center of the circle which containeth the base, to the circumference.

8. A section of a sphere containing three bases of the Dodecaedron, taketh a third part of the diameter of the said sphere.

9. The side of the Dodecaedron and the line which subtendeth the angle of the Pentagon, are equal to the right line which coupleth the middle sections of the opposite sides of the Dodecaedron.

THE THEOREMS OF ARCHIMEDES.

Concerning the Sphere and Cylinder, investigated by the *Method of indivisibles*, and briefly demonstrated by the Reverend and Learned Dr. ISAAC BARROW.

THE main Design of *Archimedes* in his *Treatise of the Sphere and Cylinder*, is to resolve these four Problems.

1. *To find the proportion of the superficies of a sphere to any determinate circle; or to find a circle equal to the superficies of a given sphere.*

2. *To find the proportion of the superficies of any segment of a sphere to any determined circle; or to find a circle equal to the superficies of any assigned segment.*

3. *To find the proportion of the sphere it self (or of its solid content) to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given sphere.*

4. To

4. *To find the proportion of a segment of a sphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given segment.*

These four Problems *Archimedes* prosecutes separately, and lays down Theorems immediately subservient to their solution; but we reduce them to two: For since an Hemisphere is the segment of a sphere, and the method of finding out its relations, in respect to the superficies and solid content, is comprehended in the general method of investigating the proportion of the segments: And from the superficies and solid content of an Hemisphere already found, the double of them, (that is the superficies and content of the whole sphere) is at the same time given. And indeed 'tis superfluous and foreign from the Laws of good Method, to investigate their relations distinctly and separately; so that if it were not a crime, I might on this account blame even *Archimedes* himself.

The whole matter therefore is reduc'd to these two Problems.

1. *To find the proportion of the superficies of any segment of a sphere to a determinate circle; or to find a circle equal to the superficies of a given segment.*

2. *To find the proportion of the solidity of any segment of a sphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to an assign'd segment of a sphere.*

I shall resolve these Problems by another much easier and shorter method: In which the order being inverted, first, I shall seek the solidity of a segment, and from thence deduce its superficies; a thing which is in my judgment well worth observing, and perform'd (as I know of) by none.

First therefore, for finding the solidity of a segment, I shall lay down two, commonly known and receiv'd, Suppositions, *viz.*

1. *That a series of magnitudes proceeding in Arithmetical Progression from nothing (inclusive) or whose common difference is equal to the least magnitude, is subduple of as many quantities equal to the greatest:* (i. e. *subduple of the product of the greatest term and number of terms:*) So that if the *sum of the terms* be called z, the *greatest term* g, and the *number of terms* n, then will

$$z = \frac{ng}{2}, \text{ or } 2z = ng.$$

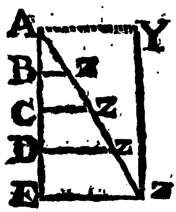
The

The truth of this Propoſition will eaſily appear by expreſſing the ſeries twice, and inverting the order;

o, a, 2a, 3a, 4a.
4a, 3a, 2a, a, o.

For ſo the difference always being equal to the leaſt quantity, 'twill be evident that each two correſpondent terms taken together are equal to the greateſt term; and alſo, that the ſeries taken twice is equal to the greateſt term repeated as many times as there are terms, *i. e.* the laſt term drawn into the number of terms.

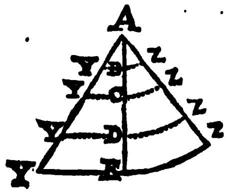
We have in a *triangle* a very clear and eaſy example of this moſt uſeful Propoſition, which is prov'd hence, *to be half a parallelogram having the ſame altitude, and ſtanding on the ſame baſe.*



Suppoſe the altitude AE of the triangle AEZ to be divided into parts indefinitely many and ſmall, AB, BC, CD, DE, and parallels BZ, CZ, DZ, EZ, drawn thro' the points of Diviſion; all theſe proceed from nothing in an Arithmetical progreſſion, and conſequently the ſum of them all, (that is, the triangle AEZ) is ſubduple of the greateſt EZ drawn into the altitude AE, by which the ſum of the terms is expreſs'd, that is ſubduple of the Parallelogram EY, whoſe baſe is EZ, and altitude AE.

But the illuſtration of the Rule will conduce more to our deſign by inferring hence, *That a circle is equal to half of the radius drawn into the circumference*, after this manner. Conceive a circle to conſiſt of as many concentric Peripheties as there are points or equal parts indefinitely many and ſmall in the radius. Theſe Peripheries, as well as their radii, proceed from the center or nothing in an Arithmetical progreſſion; and therefore their ſum, that is, the whole circle is equal to half the greateſt (or extreme circumference) drawn into the number of terms, that is, the radius.

After the same manner we may suppose the sector AEZ to consist of as many concentric Arcs BZ, CZ, DZ, EZ, as there are points (or equal parts indefinitely small) in the radius AE, which Arcs, as their radii, proceeding from a point or nothing in an Arithmetical progression, the sector also will be equal to half the radius drawn into the extreme Arc EZ. Which may be made evident also after this manner: Let us suppose the right line EY to be perpendicular to the radius AE, draw the right line AY, and from the points B, C, D, of division in the radius, draw BY, CY, DY, parallel to EY, and terminated at AY. Because EY: DY (:: *rad.* AE: *rad.* AD) :: *Arc.* EZ: *Arc.* DZ. and EY = EZ, then will DY = Arc. DZ; and in like manner will CY = CZ, and BY = BZ. Whence the triangle AEY will be = to the sector AEZ, that is, $\frac{AE \times EY}{2}$ $\left(\frac{AE \times EZ}{2}\right)$ = sector AEZ. By this means we collect that celebrated Theorem of *Archimedes, that a circle is equal to a triangle whose base is equal to the radius, and altitude equal to the periphery of the circle*; and that without any inscription or circumscription of figures, by only supposing that the Area or Superficies of the circle consists of infinitely many concentric Peripheries. Which *method of Indivisibles*, (now first of all known to me) seems no less evident (nay more evident) and perhaps less fallacious than that wherein planes are supposed to consist of parallel right lines, and solids of parallel planes; as hereafter shall be evident, when we shall collect, by this method, the proportions of spheric and cylindric superficies to one another, by knowing the solid content; and on the other hand, the solid content, by knowing the superficies, with admirable facility, and most full satisfaction in those things which are rigidly gather'd by pure *Geometry*.



Let

Let us suppose a series of quantities to proceed from 0. *(inclusive) in a duplicate Arithmetic proportion, that is,* 0, 1, 4, 9, 16, &c. *the squares of numbers in a simple Arithmetic progression,* 0, 1, 2, 3, 4, &c. *And the triple of this series will always exceed the greatest term multiplied by the number of terms; but the number of terms increasing, the proportion continually approximates, till at last it comes to an equality, when the number of terms is increased* in infinitum.

$$3 \times 0+1=3.\quad \frac{3}{2}$$
$$2 \times 1=2.$$
$$3 \times 0+1+4=15.\quad \frac{15}{12}=\frac{5}{4}$$
$$3 \times 4=12.$$
$$3 \times 0+1+4+9=42.\quad \frac{42}{36}=\frac{7}{6}$$
$$4 \times 9=36.$$
$$3 \times 0+1+4+9+16=90.\quad \frac{90}{80}=\frac{9}{8}$$
$$5 \times 16=80.$$
$$3 \times 0+1+4+9+16+25=165.\quad \frac{165}{150}=\frac{11}{10}$$
$$6 \times 25=150.$$

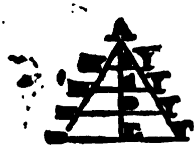
As for example, if the terms are two, the triple of the terms will be to the greatest term drawn into the number of terms as 3 to 2; if there be three terms as 5 to 4; if four, as 7 to 6; if five, as 9 to 8, and so continually: So that the antecedents of these proportions always mutually exceed one another by the number 2; and so every antecedent its consequent by 1. Whence it is evident that by how much the greater the number of terms is, by so much the more the proportion tends to equality. So 100 to 99 is less distant from the proportion of equality than 16 to 9. From hence, supposing the number of terms infinite (or infinitely great,) the triple of quantities proceeding thus in a duplicate proportion (or as the squares of the numbers, 0, 1, 2, 3, 4, *&c.* will be equal to as many quantities equal to the greatest term.

The same, as to the substance of it, is laid down by *Archimedes* in his *Book of Spirals*, as the Foundation of many Argumentations, in that, and other Books, and is well demonstrated by our learned Country-man Dr. *Wallis*: However, I thought fit to illustrate the matter by this

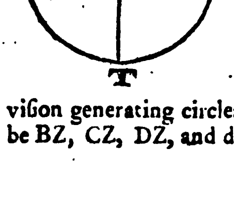
 method,

method, as being not unworthy our Consideration, and very perspicuous and intelligible in this, that 'tis free from Fractions: And by the way 'tis observ'd, that from hence we may easily find the proportion of a series triple to as many terms equal to the greatest, *viz.* as twice the number of terms less one, to twice the number of terms less two. So that if the number of the terms be 6, the proportion of a series triple to as many terms equal to the greatest will be as 11.

It will be a very easy and apt Illustration of this Rule, if we infer hence, *That a Cone is subtriple of a Cylinder, having an equal base and altitude.* For let us suppose the altitude AE of the Cone ZY to be divided into equal and indefinitely many parts, by as many parallel right lines ZY, and the lines ZY will be as the numbers 1, 2, 3, 4, &c. and the squares or circles constituted upon the diameters ZY, as 1, 4, 9, 16, &c. whence all those circles, or the whole Cone AZY (made up of the same) will be subtriple of as many circles equal to the greatest, constituted on the greatest diameter ZEY, that is, subtriple of a cylinder whose base is AEY, and altitude AE.



There occur two other most apt examples of this Rule, *viz.* by inferring, *That the complement of a Semiparabola is subtriple of a parallelogram having the same base and height*; as also, *That the space comprehended by the Spiral and Radius is subtriple of the circle in which the Spiral is generated:* But of these in another place. Wherefore to go on with what we began, these two Rules being supposed; let us conceive ZAY to be a segment of a sphere, X its center, AT its diameter, and ZAYT a great circle passing thro' the vertex, and the part AE of the Axe to be divided into an indefinitely many equal parts; and let us imagine parallel lines to be drawn thro' the points of division generating circles in the sphere, whose Radii let be BZ, CZ, DZ, and diameters ZY. I suppose the segment



ment of a sphere to consist of all these parallel circles, whose number is as great as that of the points, or equal indefinitely many small parts in the Axe AE, according to the known *Method of Indivisibles*.

But now for brevity's sake, let the diameter AT be called d, and the radius of the sphere r (if need be) and the Axe AE, by which the number of terms is express'd, call n, and one of the equal parts a; which being settled, 'tis evident, (*by the Elements*) that $\overline{BZ}^2 = AB \times BT = a \times \overline{d-a} = ad - a^2$, and in like manner $CZ^2 = AC \times CT = 2a \times \overline{d-2a} = 2ad - 4a^2$, and by the same reasoning $DZ^2 = AD \times DT = 3ad - 9a^2$, and $EZ^2 = AE \times ET = 4ad - 16a^2$, &c. that is, that the squares of the radii of the circles ZY are to one another as the rectangles, ad $2ad$, $3ad$, $4ad$, &c. (which proceed in an Arithmetical Progression from o) less by the square a^2, $4a^2$, $9a^2$, $16a^2$, &c. which go on as the squares of the numbers, 1, 2, 3, 4, &c. But by our first Rule, all the Rectangles o, ad, $2ad$, $3ad$, $4ad$, &c. are equal to half as many terms equal to the greatest AE $\times$ AT or nd, that is, $= \frac{nd \times n}{2}$

Moreover, by our second Rule, all the squares o, a^2, $4a^2$, $9a^2$, $16a^2$, &c. taken together, are equal to a third part of as many terms equal to the greatest $\overline{AE}^2$ or n^2, that is, $= \frac{n^2 \times n}{3}$.

Wherefore all the squares described upon the radii BZ, CZ, DZ, EZ, conjunctly, are equal to the difference $\frac{nnd}{2} - \frac{nnn}{3}$, (or the terms being reduc'd to the same denomination,) $\frac{3\,ndn - 2\,nnn}{6}$ and their quadruple, that is, all the squares described upon the diameters ZY, are equal to $\frac{12\,ndn - 8\,n^3}{6}$ or $\frac{6\,ndn - 4\,n^3}{3}$ whence a segment of a sphere is equal to a Cylinder, the diameter of whose base is the side of a square equal to $6\,nd - 4\,n^2$, and

and altitude is $\frac{1}{3}n$; or to a Cone having the same base, but the altitude n. or which is all one, having a base whose radius is $\sqrt{\frac{6nd - 4n^2}{4}}$ or $\sqrt{\frac{3}{2}nd - n^2}$, and altitude n as before. Which Cone we may change into a Cone upon the same base ZY with the segment ZAY, by saying, as ZE^2 (*i. e.* $dn - n^2$) to $\frac{3}{2}nd - n^2$ or (both terms being divided by n) as $d-n$ to $\frac{3}{2}d-n$, so reciprocally n to the altitude of the Cone sought: Or in the figure by making, *as TE to TE+XA, so is EA to ES. For ES will be the altitude of the Cone ZST equal to the segment of the sphere ZAT.* Which is that noted Theorem of *Archimedes*, demonstrated by him with so much labour and prolixity.

Hence, if the given segment be a *Hemisphere*, and so $n = \frac{1}{2}d$ or r, then d or $2r$ will be the altitude of a *Cone*, which having a base equal to the base of the *Hemisphere* (or to the greatest circle in the sphere) will be equal to the *Hemisphere*. And a *Cone* whose base is double of the greatest circle, and the altitude $2r$, or the *Cylinder* whose bases is $\frac{2}{3}$ of the greatest circle, and altitude $2r$. will be equal to the whole *Sphere*. Whence the whole *Sphere* is $\frac{2}{3}$ of a *Cylinder*, the diameter of whose base is $2r$, and the altitude also $2r$. And this is the chief Theorem of *Archimedes*, viz. *That a sphere subsesquialter or $\frac{2}{3}$ of that Cylinder, whose Altitude and Diameter of the base is equal to the Diameter of the Sphere.*

Furthermore, not to pass over any thing in our Author which seems to be to our purpose:

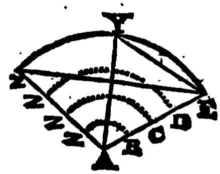
If to the sum first found, representing a segment, *viz.* $\frac{6ndn - 4nnn}{3}$ we add $\frac{2d^2n - 6dn^2 + 4n^3}{3}$ ($= \frac{4dn - 4n^2}{3} \times \frac{d - 2n}{2} = \frac{4}{3}ZE^2 \times XE$) representing the Cone ZXY, the aggregate $\frac{2}{3}d\ d\ n$ will represent the Sector of the Sphere ZXYA, which for that reason will be equal to a Cylinder, the diameter of whose base $\sqrt{dn}$, and the altitude $\frac{2}{3}d$, or to a Cone, the diameter of whose base is $\sqrt{dn}$, and the altitude $2d$, or also to a Cone, the Radius of whose base is $\sqrt{dn}$, and the altitude $\frac{1}{2}d = r$ (it being reciprocally as $4dn : dn :: 2d : \frac{1}{2}d$,) that is, to a Cone, the Radius of whose base is the Line AZ, drawn

from

from the vertex to the circumference of the base of the segment, (for $AZ^2 = TA \times AE = dn$,) and the altitude r.

And this is the next famous Theorem of *Archimedes*, concerning the solidity of the sector of the Sphere. *viz. That the sector of a sphere is equal to a Cone, whose base is a circle described by a Radius equal to a line drawn from the vertex to the circumference of the base of the segment, and whose altitude is equal to the Radius of the sphere.*

And thus I think I have compleated that which belongs to the solidity of a sphere, and its parts, with sufficient brevity and perspicuity. From hence we shall deduce the Resolution of the other Problem, which I proposed concerning the surface of the segment of a sphere; and then of the whole sphere. To obtain this, as we supposed before, a *Circle* to consist of concentric Peripheries, and the *Sector of a Circle* of concentric Arcs, (in the number of which, the greatest, and the least, or a point is reckon'd: So now we suppose spheres to consist of concentric spherical superficies, and the *Sectors of Spheres* of like concentric superficies; as for example, the sector of the sphere ZAE, of the superficies BZ, CZ, DZ, EZ, &c.) which supposition indeed seems so easy and natural, that in my judgment 'tis sufficient only to propose it; neither is a further explication wanting to gain an assent to it.

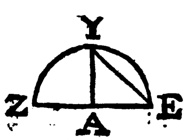


2. We suppose these spherical superficies to be in a duplicate Ratio of the Radius of the spheres: This is the common affection of all like superficies, and it seems to agree very well with the superficies of spheres, because they appear to be most uniform and similar. But this Supposition might easily be evinc'd and establish'd by the same sort of arguing, as spheres are proved to be in triplicate proportion to their Diameters or Radii; or might have been join'd as a *Corollary* to *Prop.* 17. and 18. *Elem.* 12. where the superficies of like Polygones are suppos'd to be inscribed in spheres, having as well the superficies in a duplicate, as the solidity in a triplicate Ratio of the Diameters of the Spheres. These things being premis'd, let us suppose AE a Radius, or the side of the Sector of a Sphere EAZ, to be divided into equal and indefinitely many small parts, and the sector AEZ to consist of these spheri-

cal superficies BZ, CZ, DZ, EZ, it will be evident that all those superficies in the Progression are as the squares of the Radii, that is, as $\overline{AB}^2$, $\overline{AC}^2$, $\overline{AD}^2$, $\overline{AE}^2$, &c. or as the squares of the numbers 1, 2, 3, 4, &c. whence by our second Rule, the sum of all these superficies, that is, the sector AEZ, will be $\frac{1}{3}$ of as many superficies equal to the greatest EZ, that is, $\frac{1}{3}$ of the greatest EZ, drawn into r the number of terms. Whence a sector is equal to a Cylinder, whose base is $\frac{1}{3}$ of the greatest or extreme superficies of the sector, and whose altitude is r: Or to a Cone whose base is equal to the superficies of the sector, and its altitude r, which is the last of *Lib.* 1. but we just now prov'd that a sector is equal to a Cone whose altitude is r, and base a circle, describ'd by the Radius YE, drawn from the vertex of the segment EYZ to the circumference of the base. Wherefore a Cone, whose altitude is r, and base equal to the superficies of the sector, is equal to a Cone of the same altitude, whose base is a circle describ'd by the Radius YE.

And so the superficies of the sector EYZ is equal to a circle describ'd by the Radius YE. Which certainly is the principal Theorem of all those that occur in the Books of *Archimedes*, nor is there found a more excellent one in all Geometry; *viz. That the superfixes of any segment of a sphere is equal to a circle whose Radius is a right line drawn from the vertex of the segment to the circumference of the bases:* And hence, *that the superficies of an Hemisphere is double to the base, or equal to two great circles of the sphere.*

For in this Case $\overline{YE}^2 = \overline{AZ}^2 + \overline{AY}^2 = 2\ \overline{AE}^2$, and consequently a circle described by the Radius YE is equal to two circles describ'd by the Radius AE. Whence also, *the superficies of the whole sphere is quadruple, a circle having the same Radius with the sphere*, that is, *quadruple the greatest circle in the sphere; and equal to a circle whose Radius is the diameter of the sphere.* From hence it follows, *that the superficies of a sphere is equal to the superficies of a Cylinder of the same height and breadth*; for the superficies of that Cylinder is quadruple to the base, as we shall shew hereafter. And these are the most noted Theorems of *Archimedes*. Nay, from hence



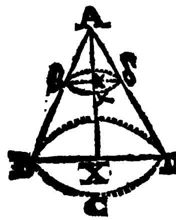
all

all those things follow, which he has written concerning the superficies of spheres, and their segments. So that from these few and easy Suppositions, I have demonstrated whatever seems to be of any Note in the Books *of the Sphere and Cylinder*.

I will only add, that after by the method of *Archimedes*, (for I think scarce any other can be invented, besides ours, for finding the solidity) the superficies of segments are found equal to the circle described by the Radii YE; hence it will plainly follow, that the superficies of spheres, and thence of like sectors are in a duplicate ratio of the Radii of the spheres; and consequently from the superficies thus found, the contents of segments, and of whole spheres may be mutually deduced, and that very clearly and expeditiously, after this manner. Because in the sector EAZ (fig. Pag. 517.) the superficies BZ, CZ, DZ, EZ, proceed as the squares describ'd upon AB, AC, AD, AE, that is, as 1, 4, 9, 16, &c. the whole sector will be equal to $\frac{1}{3}$ of as many superficies equal to the greatest EZ, or $\frac{1}{3}$ EZ $\times$ *r*, that is, to a Cylinder whose base is $\frac{1}{3}$ EZ, and altitude *r*, or to a Cone whose base is EZ, and altitude *r*. But EZ is supposed equal to a circle whose Radius is YE, wherefore the sector EAZ is equal to a Cone whose base is a circle described by the Radius YZ and altitude *r*: Which is *Archimedes*'s universal Theorem for the contents of *Sectors*. Whence if from this the Cone ZAE standing on the base of the segment EYZ, and having the vertex at the center of the sphere A, be subducted, you'll have that segment EYZ. But when the sector EYZ is a Hemisphere, there will be no such Cone to be subducted; and for that reason a Cylinder whose base is $\frac{2}{3}$ EZ, and altitude *r*, or the Cone whose base is 2EZ, and altitude likewise *r*, will be equal to the whole sphere. But the Superficies of the Hemisphere EZ, is proved to be equal to two of the greatest circles in the sphere, whence the whole sphere is given. This is *Archimedes*'s first and principal Theorem, for the content of a sphere; whence 'tis easily deduced, that a sphere is $\frac{2}{3}$ of a circumscrib'd Cylinder, that is, of a Cylinder whose altitude and diameter of its base is equal to the diameter of the sphere.

The Doctrine of our author [*Archimedes*] seems to make against, and subvert the new and celebrated *Method of indivisibles*, and is press'd to that end by *Tacquet*; for

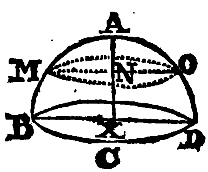
for instance, (*Prop.* 2. *lib.* 2. *Cylindr.*) For the usual process of that method seems to exhibit the dimension of the superficies of a Cone, (as also of a sphere, and of other Curves) different enough from what our Author and others have demonstrated: As for example, let us suppose ABCD a right Cone, whose Axe is AX, and base BCD, and plane $\beta\chi\delta$ drawn, at pleasure, parallel to the base BCD. And since, as *Diam.* BD : *Periph.* BCD :: *Diam.* $\beta\delta$: *Periph.* $\beta\chi\delta$, and so every where it will be (according to the *Method of Indivisibles*, and by 12. 5.) as *Diam.* BD, to *Periph.* BCD, so is the triangle ABD, consisting of those parallel Diameters, to the *Conic Superficies* ABCD, consisting of those Peripheries, *i. e. Diam.* BD : *Periph.* BCD :: $\frac{AX \times BD}{2} : \frac{AX \times \textit{Periph.}\ BCD}{2}$ Whence $\frac{AX \times \textit{Periph.}\ BCD}{2}$ will be equal to the superficies of the Cone; which is false and contrary to what was demonstrated just now. For we demonstrated that the superficies of the Cone was $\frac{AB \times \textit{Periph.}\ BCD}{2}$.



In answering this Objection, we say, that the *Method of Indivisibles*, in the speculation of Perimeters, and of Curve Surfaces, proceeds otherwise than in the speculation of plane Surfaces and solid Contents. It does indeed suppose that the Area of plane Figures consists as it were of parallel right lines, and the contents of solids of parallel Planes, and that their number may be express'd by the altitude of the Figures: But it by no means supposes, that the Perimeters of plane figures consist of points, or the superficies of solids of lines, the number of which may be express'd by the altitude of the figure. As for example, altho' the triangle ABD (in the last figure) consists of lines parallel to BD, the number of which is expressed by the number of points in the perpendicular AX, that is, by the length of the

the perpendicular: Yet it would be absurd to suppose that the line AB consists of points, whose number may be express'd by the number of points in a less line AX. For altho' the right lines $\beta\delta$ drawn thro' each infinitely small part of AX, divide AB into as many infinitely small parts, yet those parts are not of the same Denomination or Quality with the parts of AX, but somewhat greater than them; so that if the parts of AX be look'd upon as points, the parts of AB are not to be called points, but greater than points; and on the contrary, if the parts of AB be called points, the parts of AX are to be look'd upon as less than points, if it be lawful to speak so. For the points which are treated of in the *Method of Indivisibles* are not absolutely points, but indefinitely small parts, which usurp the name of points, because of their affinity to them. Since therefore points don't admit of greater and less, the name of points is not at the same time to be attributed to the parts of different magnitudes; consequently, tho' the number of the greater parts of AB may be express'd by the number of the lesser parts of AX, yet the number of points in AB can no ways be expressed by the number of points in AX, (that is, by the number of parts in AX, equal to the number of parts in AB, which are called points.) The line AB has as many points as there are in it self alone, or another line equal to it self, nor can it be determin'd by any other measure. After the same manner, this method don't suppose the conic Surface ABCD to consist of as many parallel circumferences perpetually increasing from the vertex A, or decreasing from the base BD, as there are points in the Axe AX, but rather of as many thus increasing or decreasing as there are points in the side AB. For in the Revolution of the line AB about the Axis AX, (whereby the superficies of the Cone is generated) every point in the line AB produces a circumference, and consequently more circumferences are produced than the points contained in the Axis AX. Therefore if you would extend the *Method of Indivisibles* to the superficies of solids, and suppose those superficies to consist of parallel lines, you ought not to compute this by the parallel Areas constituting the solid, that is, not to number those Areas by the altitude of the solid, but by other lines agreeable to the

the condition of each figure. Which lines, in figures that are not irregular, may easily be determin'd: For instance, in the equilateral Pyramid ABCD, whose Axe is AX, supposing that the lateral surface of the pyramid consists of Perimeters of triangles, parallel to the base BCD, these can neither be computed by the altitude AX, nor by the side AB, (for by the former the thing requir'd would be wanting of the true Dimension, and by the latter 'twould exceed it) but by the line AE drawn from the vertex A perpendicular to the side BC of the base: The reason of which is, that every plane side of a Pyramid, as ABC, consists of parallel right lines computed by the altitude AE. After the same manner, supposing that the superficies of the Hemisphere BAD, consists of Peripheries of circles parallel to the base BCD, the number of them is not to be computed by the Axis AX, but by the Quadrantal Arc AB, because that every point of the Arc AD in revolving produces a circumference; and so any superficies, whether plane or curv'd, which is conceived to consist of equidistant right or curv'd lines, is to be computed by a line cutting those equidistant lines perpendicularly. For since those equidistant lines, in this *Method of Indivisibles*, are not consider'd absolutely as lines having an infinitely small breadth, which is the same with the breadth or thickness of the point describing those equidistant lines in their Circumvolution, and since the same equidistant lines divide the line cutting them perpendicularly into parts measuring its breadth, those parts are to be look'd upon as such sort of points, and consequently the number of equidistant lines, or the sum of those breadths is to be computed by the number of points in the line cutting them perpendicularly, that is, by the length of that line, and not by a line of any other

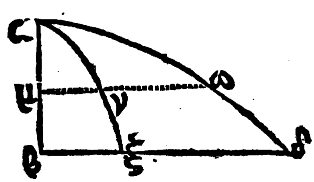


ther length, for that will consist of more or less points.

Hence therefore in the speculation of the superficies of solids, the *Method of Indivisibles* is not unuseful, but rather very commodious, provided it be rightly understood, and applied according to the Rule prescrib'd. For by the help of it even these superficies may be found, if so be we have some convenient *Data* presuppos'd, on which the reasoning may be founded: For instance, we might, by the help of it, investigate the superficies of a Cone, by reasoning after this manner.

If the superficies of the Cone ABC (fig. pag. 510.) be divided into innumerable Peripheries of circles $\beta\chi\delta$ parallel to the base BCD, the breadth of those Peripheries taken together, make up the side AB cutting them perpendicularly, and consequently there will be as many Peripheries as there are points in the line AB, that is, their number may be express'd by the number of points in AB, or by its length. Wherefore if you draw perpendiculars equal to the Peripheries to every point of AB, a superficies will be made out of those perpendiculars equal to the superficies of the Cone; but that superficies will be a triangle whose height is AB, and base equal to the greatest Periphery BDC, and so the superficies of the Cone will be $= \frac{1}{2}$ AB $\times$ *Periph.* BDC, which conclusion agrees with the things laid down and demonstrated by *Archimedes*.

After the same manner, if you take any right line $\alpha\beta$ equal to the quadrantal Arc AB of the Hemisphere (*in pag.* 512.) and to each of its points μ let the perpendiculars $\mu\nu$ be erected equal to the Radii MN of the parallel circles MOM passing thro' the corresponding points M of that quadrantal Arc, the greatest of which $\beta\xi$ let be equal to the Radius BX of the base of the Hemisphere: The figure $\alpha\beta\xi$ will contain the Radii of all the circles of



whose